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# Antimagicness of subdivided fans

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**Abstract:** A graph $\Gamma$ (simple, finite, undirected) with an $\Omega$-covering has an $(\alpha,\delta)$-$\Omega$-antimagic labeling if the weights of all subgraphs $\Omega$ of graph $\Gamma$ constitute an arithmetic progression with the common difference $\delta$. Such a graph is called *super* $(\alpha,\delta)$-$\Omega$-antimagic if $\nu(V(\Gamma)) = \{1,2,3,\dots,|V(\Gamma)|\}$. In the present paper, the cycle coverings of subdivision of fan graphs has been considered and results are proved for several differences.

**Keywords:** $\Omega$-covering, super $(\alpha,\delta)$-$\Omega$-antimagic graph, cycle-antimagic, super cycle-antimagic, fan graphs.

**MSC:** 05C78, 05C70.

## 1. Introduction

Let $\Gamma = (V(\Gamma), E(\Gamma))$ be a finite simple and undirected graph with a family of subgraphs $\Omega_1, \Omega_2, \dots, \Omega_t$ such that every element of $E(\Gamma)$ belongs to $\Omega_i \cong \Omega$, $i = 1,2,\dots,t$, then $\Gamma$ admits an $\Omega$-*covering*. An $\Omega$-covered graph $\Gamma$ with $\nu$ is called an $(\alpha,\delta)$-$\Omega$-antimagic if $wt_\nu(\Omega) = \{\alpha, \alpha+\delta, \dots, \alpha+(t-1)\delta\}$ where the associated $\Omega$-*weights* denoted by $wt_\nu(\Omega)$ are defined as

$$wt_\nu(\Omega) = \sum_{v \in V(\Omega)} \nu(v) + \sum_{e \in E(\Omega)} \nu(e).$$

and $\alpha > 0$ and $\delta \geq 0$ are two integers, $t$ is the number of $\Omega_i \cong \Omega$. For a total labeling $\nu$ to be super we require $\nu(V(\Gamma)) = \{1,2,\dots,|V(\Gamma)|\}$.

The results about $\Omega$-(super)magic graphs with $\Omega$ as cycle, path and tree can be studied in [1–7].

Inayah *et al.* [8] introduced the $(\alpha,\delta)$-$\Omega$-antimagic labeling. We refer [9–11] for some results on super $(\alpha,\delta)$-$\Omega$-antimagic labeling. In [12], Lih proved that $F_n$ is $C_3$-supermagic for every $n$ except $n \equiv 2 \pmod 4$. In [7], Ngurah *et al.* proved that $F_n$ is $C_3$-supermagic for every $n \geq 2$. In the present paper, we proved the super $(\alpha,\delta)$-$C_{r+2k+3}$-antimagic labelings of subdivided fans for differences $\delta = 0,1,2,3,4$.

## 2. Preliminaries

In this section, we give basic definitions of concepts concerning a subdivided fan $F_n(r,k)$.

**Definition 1.** A graph $F_n \cong P_n + K_1$ is called *fan* graph obtained by the join of *path* $P_n$ and one isolated vertex $K_1$.

The *central vertex*, or the *hub vertex* is of degree $n$ and *path vertices* are the other ones. *Spokes* are the adjacent edges of central vertex and *path edges* are the remaining edges.

$$\begin{aligned} V(F_n) &= \{c\} \cup \{x_1, x_2, \dots, x_n\}, \\ E(F_n) &= \{x_1x_2, x_2x_3, \dots, x_{n-1}x_n\} \cup \{cx_1, cx_2, \dots, cx_n\}. \end{aligned}$$

**Definition 2.** The subdivided fan $F_n(r,k)$ is the graph obtained from a fan $F_n$ by inserting $r \geq 1$ new vertices $\{v_1^{(i)}, \ldots, v_r^{(i)}\}$ into each path edge $x_i x_{i+1}, 1 \leq i \leq n-1$, denoted by $P_{x_i x_{i+1}}$-vertices and by inserting $k \geq 1$ new vertices $\{w_1^{(i)}, \ldots, w_k^{(i)}\}$ into every spoke $cx_i, 1 \leq i \leq n$, denoted by $S^{(i)}$-vertices.

$$
\begin{aligned}
E(P_{x_i x_{i+1}}) &= \{x_i v_1^{(i)}, v_2^{(i)} v_3^{(i)}, \ldots, v_{r-1}^{(i)} v_r^{(i)}, v_r^{(i)} x_{i+1}, 1 \leq i \leq n-1\}, \\
E(S^{(i)}) &= \{cw_1^{(i)}, w_2^{(i)} w_3^{(i)}, \ldots, w_{k-1}^{(i)} w_k^{(i)}, w_k^{(i)} x_i, 1 \leq i \leq n\}.
\end{aligned}
$$

Let $C_{r+2k+3}^{(i)}$ be the $i^{\text{th}}$-subcycle. For the weight of $i^{\text{th}}$-subcycle $C_{r+2k+3}^{(i)}$, we obtain

$$
\begin{aligned}
wt_\psi(C_{r+2k+3}^{(i)}) &= \sum_{u \in V(C_{r+2k+3}^{(i)})} \psi(u) + \sum_{e \in E(C_{r+2k+3}^{(i)})} \psi(e) \\
&= \left( \psi(x_i) + \psi(x_{i+1}) + \psi(c) + \sum_{v \in V(P_{x_i x_{i+1}})} \psi(v) + \sum_{w \in V(S^{(i)})} \psi(w) \right) \\
&\quad + \left( \sum_{e \in E(P_{x_i x_{i+1}})} \psi(e) + \sum_{e \in E(S^{(i)})} \psi(e) + \sum_{e \in E(S^{(i+1)})} \psi(e) \right). \qquad (1)
\end{aligned}
$$

where indices $i$ are taken modulo $n$.

## 3. Main results

In this section, we introduce the super $(\alpha, \delta)$-$C_{r+2k+3}$-antimagic labelings of subdivided fans for differences $d = 0, 1, 2, 3, 4$.

**Theorem 1.** *Let $r, k \geq 1$ and $n \geq 3$ be positive integers. The subdivided fan $F_n(r,k)$ is super $(\alpha, \delta)$-$C_{r+2k+3}$-antimagic for difference $\delta = 0, 1, 4$.*

**Proof.** The total labeling $\psi_\delta$ is defined as:

$$\psi_{\{\delta\}}(c) = 1$$

$$
\psi_{\{0,4\}}(x_i) = \begin{cases} \lceil \frac{n}{2} \rceil + 2 - \frac{i+1}{2}, & if\ i \equiv 1 \pmod 2 \\ n + 2 - \frac{i}{2}, & if\ i \equiv 2 \pmod 2 \end{cases}
$$

$$
\psi_{\{1\}}(x_i) = \begin{cases} 1 + \frac{i+1}{2}, & if\ i \equiv 1 \pmod 2 \\ 1 + \lceil \frac{n}{2} \rceil + \frac{i}{2}, & if\ i \equiv 2 \pmod 2 \end{cases}
$$

$$
\psi_{\{0,1\}}(cw_1^{(i)}) = \begin{cases} 2(n-1)(r+1) + 2(nk+1) + \frac{i+1}{2}, & if\ i \equiv 1 \pmod 2 \\ 2(n-1)(r+1) + 2(nk+1) + \lceil \frac{n}{2} \rceil + \frac{i}{2}, & if\ i \equiv 2 \pmod 2 \end{cases}
$$

$$\psi_{\{4\}}(cw_1^{(i)}) = 2n(r+k) + (3n - 2r + 1) - i.$$

For $\delta = 0, 1, 4$

$$
\begin{aligned}
\psi_\delta(V(P_{x_i x_{i+1}})) &= \{(n-1)j + 2 + i : 1 \leq i \leq n-1, 1 \leq j \leq r\} \\
\psi_\delta(E(P_{x_i x_{i+1}})) &= \{(n-1)(2r + 2 - j) + n(k+1) + 2 - i : 1 \leq j \leq r+1\} \\
\psi_\delta(V(S^{(i)})) &= \{r(n-1) + 1 + nj + i : 1 \leq i \leq n, 1 \leq j \leq k\} \\
\psi_\delta(E(S^{(i)}) \setminus \{cw_1^{(i)}\}) &= \{2n(r+k) + (3n - 2r + 1) - nj - i : 1 \leq j \leq k\}
\end{aligned}
$$

where indices $i$ are taken modulo $n$. Evidently $\psi_\delta$ is a super labeling as $V(F_n(rk))) = \{1, 2, \ldots, n(k+r+1) - r + 1\}$. The spoke vertices are labeled with the numbers $n+2, n+3, \ldots, n+2k+1$ and the path edge vertices are labeled with $n+2k+2, n+2k+3, \ldots, n(k+r+1) - r + 1$. Clearly,

$$\sum \left( \psi_\delta(V(S^{(i)})) + \psi_\delta(E(S^{(i)}) \setminus \{cw_1^{(i)}\}) \right) = 3rk(n-1) + k(3n+2nk+2)$$
$$\sum \left( \psi_\delta(V(P_{x_i x_{i+1}})) + \psi_\delta(E(P_{x_i x_{i+1}})) \right) = nr(2r+k+4) + n(k+2) + +r(1-2r) + 1 - i \quad (2)$$

According to (1) and (2), we obtain

$$\begin{aligned} wt_{\psi_0}(C^{(i)}_{r+2k+3}) &= 4(n-1)(r+1) + 4(nk+1) + 6rk(n-1) + 2\left\lceil \frac{n}{2} \right\rceil + n + 3 + 2k(3n+2nk+2) \\ &\quad + nr(2r+k+4) + n(k+2) + r(1-2r) + 1 \\ wt_{\psi_0}(C^{(i)}_{r+2k+3}) &= n + 2\left\lceil \frac{n}{2} \right\rceil + nk(7r+4k+11) + 2nr(r+4) + r(1-2r-6k) + 2(3n+2k-2r) + 4. \quad (3) \end{aligned}$$

Equation (3) shows that all $C^{(i)}_{r+2k+3}$-weights are independent of $i$.
According to (1) and (2), we obtain

$$\begin{aligned} wt_{\psi_1}(C^{(i)}_{r+2k+3}) &= 4(n-1)(r+1) + 4(nk+1) + 6rk(n-1) + 2\left\lceil \frac{n}{2} \right\rceil + 5 + 2i + 2k(3n+2nk+2) \\ &\quad + nr(2r+k+4) + n(k+2) + r(1-2r) + 1 - i \\ wt_{\psi_1}(C^{(i)}_{r+2k+3}) &= 2\left\lceil \frac{n}{2} \right\rceil + nk(7r+4k+11) + 2nr(r+4) + r(1-2r-6k) + 2(3n+2k-2r) + 6 + i. \quad (4) \end{aligned}$$

Equation (4) shows that all $C^{(i)}_{r+2k+3}$-weight consists of consecutive integers.
According to (1) and (2), we obtain

$$\begin{aligned} wt_{\psi_4}(C^{(i)}_{r+2k+3}) &= 4(n-1)(r+1) + 4(nk+1) + 6rk(n-1) + 2\left\lceil \frac{n}{2} \right\rceil + 5 + 2i + 2k(3n+2nk+2) \\ &\quad + nr(2r+k+4) + n(k+2) + r(1-2r) + 1 - i \\ wt_{\psi_4}(C^{(i)}_{r+2k+3}) &= n + \left\lceil \frac{n}{2} \right\rceil + r(1-2r) + k(4nk-5r+9) + 2n(3k+4)(r+1) - 6n + 7 - 4i. \quad (5) \end{aligned}$$

Equation (5) shows that all $C^{(i)}_{r+2k+3}$-weight constitute an arithmetic progression with common difference $\delta = 4$. This completes the proof. □

**Theorem 2.** *Let $r, k \geq 1$ and $n \geq 3$ be positive integers. The subdivided fan $F_n(r,k)$ is super $(\alpha, \delta)$-$C_{r+2k+3}$-antimagic for difference $\delta = 2, 3, 5$.*

**Proof.** The total labeling $\psi_\delta$ is defined as:

$$\psi_{\{\delta\}}(c) = 1$$
$$\psi_{\{\delta\}}(x_i) = 2i$$
$$\psi_{\{\delta\}}(v_r) = 2i + 1$$

$$\psi_{\{2\}}(cw_1^{(i)}) = 2n(r+k) + (3n-2r+1) - i$$

$$\psi_{\{3\}}(cw_1^{(i)}) = \begin{cases} 2n(r+k) + (3n-2r+1) - \frac{i+1}{2}, & \text{if } i \equiv 1 \pmod 2 \\ 2n(r+k) + (3n-2r+1) - \lceil \frac{n}{2} \rceil - \frac{i}{2}, & \text{if } i \equiv 2 \pmod 2 \end{cases}$$

$$\psi_{\{5\}}(cw_1^{(i)}) = \begin{cases} 2\{r(n-1) + n(k+1)\} + \frac{i+1}{2}, & \text{if } i \equiv 1 \pmod 2 \\ 2\{r(n-1) + n(k+1)\} + \lceil \frac{n}{2} \rceil + \frac{i}{2}, & \text{if } i \equiv 2 \pmod 2 \end{cases}$$

For $\delta = 2, 3, 5$

$$\begin{aligned} \psi_\delta(V(P_{x_i x_{i+1}})) &= \{n + (n-1)j + 1 + i : 1 \leq i \leq n-1, 1 \leq j \leq r-1\} \\ \psi_\delta(E(P_{x_i x_{i+1}})) &= \{(n-1)(2r+2-j) + n(k+1) + 2 - i : 1 \leq j \leq r+1\} \\ \psi_\delta(V(S^{(i)})) &= \{r(n-1) + 1 + nj + i : 1 \leq i \leq n, 1 \leq j \leq k\} \\ \psi_\delta(E(S^{(i)}) \setminus \{cw_1^{(i)}\}) &= \{2n(r+k) + (3n-2r+1) - nj - i : 1 \leq j \leq k\} \end{aligned}$$

where indices $i$ are taken modulo $n$.

Evidently $\psi_\delta$ is a super labeling as $V(F_n(rk))) = \{1, 2, \dots, n(k+r+1) - r + 1\}$. The spoke vertices are labeled with the numbers $n+2, n+3, \dots, n+2k+1$ and the path edge vertices are labeled with $n+2k+2, n+2k+3, \dots, n(k+r+1) - r + 1$. Clearly,

$$\sum \left(\psi_\delta(V(S^{(i)})) + \psi_\delta(E(S^{(i)}) \setminus \{cw_1^{(i)}\})\right) = 3rk(n-1) + k(3n+2nk+2)$$
$$\sum \left(\psi_\delta(V(P_{x_i x_{i+1}})) + \psi_\delta(E(P_{x_i x_{i+1}}))\right) = n(r+1)(k+3) + 2n(r^2-1) + r(n-2r+1) + 1. \quad (6)$$

According to (1) and (6), we obtain

$$wt_{\psi_2}(C^{(i)}_{r+2k+3}) = 2i + 3 + n\{k(r+5) + 7r + 3\} + 2nk(2k+3) + 2n(r^2-1) + 2(3n - 2r + 2k + 1)$$
$$wt_{\psi_2}(C^{(i)}_{r+2k+3}) = nk(4k+r+11) + n(2r^2+7r+1) + 2(3n-2r+2k) + 5 + 2i. \quad (7)$$

Equation (7) shows that all $C^{(i)}_{r+2k+3}$-weights constitute an arithmetic progression with common difference $\delta = 2$.

According to (1) and (6), we obtain

$$wt_{\psi_3}(C^{(i)}_{r+2k+3}) = 6n - 4r + 5 + 3i - \left\lceil \frac{n}{2} \right\rceil + r(n-2r+1) + 6kr(n-1)$$
$$+2kn(2k+3) + 4k + n(2r^2+5k+7r+rk+1)$$
$$wt_{\psi_3}(C^{(i)}_{r+2k+3}) = 2nr(r+4) + nk(4k+r+11) + 6rk(n-1) - r(2r+3) + 7n + 4k - \left\lceil \frac{n}{2} \right\rceil + 5 + 3i. \quad (8)$$

Equation (8) shows that all $C^{(i)}_{r+2k+3}$-weights constitute an arithmetic progression with common difference $\delta = 3$.

According to (1) and (6), we obtain

$$wt_{\psi_5}(C^{(i)}_{r+2k+3}) = nk(6r+11) + nr(3r+8) - r(2r+3) + 2k(2nk - 3r + 2) + 5n + \lceil \frac{n}{2} \rceil + 5 + 5i. \quad (9)$$

Equation (9) shows that all $C^{(i)}_{r+2k+3}$-weights constitute an arithmetic progression with common difference $\delta = 5$. □

## 4. Concluding remarks

An $\Omega$-covering graphs is the extension of the edge-antimagic labeling and generalizes the structure for $\Omega$-antimagic labeling. Several results concerning $\Omega$-antimagic labelings for different families of graphs are proved and available in literature. In the present manuscript, the super $(\alpha, \delta)$-$C_n$-antimagicness of subdivided fans has been considered for few of differences. One can work to extend the labeling for further differences greater than 5.

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