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Max-plus Algebra and Application to Matrix Operations

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Abstract

This paper study mathematical theory, called the max-plus algebra, which have the wherewithal for a uniform treatment of most problems that arise in the area of Operations Research. The basic properties of max-plus algebra is also explained including how to solve systems of max-plus equations.

In this paper, the discrepancy method of max-plus is used to solve $n \times n$ and $m \times n$ system of linear equations where $m \leq n$. From the examples presented, it is clear that an $n \times n$ system of linear equations in $(\mathbb{R}_{max}, \oplus, \otimes)$ and $(\mathbb{R}, +, \cdot)$ either had One solution, an Infinite number of solutions or No solution. Also, both $m \times n$ system of linear equations (where $m < n$) in $(\mathbb{R}_{max}, \oplus, \otimes)$ and $(\mathbb{R}, +, \cdot)$ have either an infinite number of solutions or no solution. It is therefore clear that many charateristics of the max-plus algebraic structure can be likened to the conventional mathematical structures. Max-plus is used to solve different types of matrix operations.

We also applied max-plus algebra in solving linear programming problem involving linear equations and inequalities.

Keywords: Max-plus algebra; matrix operations.

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1 Introduction

Max-plus algebra is defined as the algebraic structure in which simple addition and multiplication are replaced by $\tau \oplus \psi = \max(\tau, \psi)$ and $\tau \otimes \psi = \tau + \psi$, respectively. Each system of linear equations in max-plus algebra can be written in the matrix form $B \otimes x = \psi$, where B is a matrix and ψ is a vector of suitable size. The max-plus algebra emerged in the late 1950's, soon after the area of Operations Research started to develop [1]. This algebraic structure (Max-plus algbra) is a semi-ring whose elements are the typical real numbers in addition to $-\infty$, where the operator of multiplication, $\otimes$, represents standard addition and the operator of addition, $\oplus$, represents taking a maximum of the real numbers been added [2, 3]. Because there is no additive inverse in the max-plus algebra, problem formulation and solutions require different techniques [4].

2 Algebraic Characteristics of Max-Plus Algebra

The max-plus algebra is an algebraic structure made up of real numbers where the traditional operations of multiplication is substituted by the operation of standard addition and addition is substituted by the operation of taking a maximum of the real numbers [5]. Indubitably, for all $\tau, \psi, \gamma \in \mathbb{R}_{max}$, the operations $\otimes$ and $\oplus$ can be defined in max-plus algebra as $\tau \otimes \psi = \tau + \psi$ and $\tau \oplus \psi = max(\tau, \psi)$ respectively. For instance, $5 \otimes 3 = 5 + 3 = 8 = 3 + 5 = 3 \otimes 5$ and $7 \oplus 5 = max(7, 5) = 7 = max(5, 7) = 5 \oplus 7$.

In max-plus algebra $\epsilon = -\infty$ is the additive identity: $\tau \oplus \epsilon = \epsilon \oplus \tau = max(\tau, -\infty) = \tau$, for $\tau \in \mathbb{R}_{max}$.

The multiplicative identity is $e = 0$: $\tau \otimes e = e \otimes \tau = \tau + 0 = \tau$, for all $\tau \in \mathbb{R}_{max}$.

Also the distributive property also exist in max-plus algebra, that is: $\tau \otimes (\psi \oplus \gamma) = \tau + max(\psi, \gamma) = max(\tau + \psi, \tau + \gamma) = (\tau \otimes \psi) \oplus (\tau \otimes \gamma)$. This shows that $\otimes$ is distributive over $\oplus$, [6].

The above obviously proves that $\otimes$ and $\oplus$ are commutative and do comply with other properties comparable to the traditional $\times$ and $+$ in algebra, [6]. Other properties of max-plus algebra are:

- Multiplicative inverse, if $\tau \neq \epsilon$ then $\exists$ a distinct ψ with $\tau \otimes \psi = e$.
- Unit Element, $\tau \otimes e = e \otimes \tau = \tau$
- $\tau \otimes (-\infty) = \tau + (-\infty) = -\infty$. Hence the additive identity, ϵ, is absorbing under multiplication, thus for $\tau \in \mathbb{R}_{max}, -\infty \otimes \tau = -\infty = \tau \otimes (-\infty)$.
- Undoubtedly, the operation of taking a maximum is commutative and associative [7], therefore $(\mathbb{R}_{max}, \oplus)$ is an abelian semi-group [8]. $(\mathbb{R}_{max}, \oplus)$ is not a group, because $\tau \in (\mathbb{R}_{max}$ has an additive inverse iff $\tau = -\infty$
- The solution to $\tau \oplus \epsilon = \psi$ is $\epsilon = \psi$ *iff* $\psi \geq \tau$. If $\psi = \tau$, it follows that the solution for ϵ can be any number equal to or less than ψ, and $\tau + \epsilon = \psi$ has no solution if $\psi \leq \tau$. The system $\tau + \epsilon = -\infty$ has a solution only if $\tau = -\infty$. For the reason that $\tau + \tau = \tau$, each element of $(\mathbb{R}_{max})$ is idempotent with regard to $\oplus$.
- The existence of a zero element $\tau \oplus \epsilon = \epsilon \oplus \tau = \tau$.

3 Matrices in Max-Plus Algebra

Max-plus algebra can be used in matrices [9]. Matrix addition in max-plus can only be performed on matrices of the same dimensions [10]. The results from the matrix sum $A \oplus B$ is the maximum from the corresponding entries. Whilst the scalar multiplication of a matrix in max-plus is where each entry of the matrix is increased by the scalar.

A zero matrix is a matrix that has all the entries being $-\infty$, which is being denoted by $-\infty$.

An identity matrix has its diagonal as a 0 and the other entries being $-\infty$. It is denoted by 0. For any matrix B and I of the same dimensions $I \otimes B = B \otimes I$

Suppose $X = [\tau_{ij}], Y = [\psi_{ij}]$, and $Z = [\gamma_{ij}]$ be $m \times n$ matrices with elements in $(\mathbb{R}_{\max})$ and $q \in (\mathbb{R}_{\max})$, then:

$X \oplus Y = [\tau_{ij} \oplus \psi_{ij}] = [max(\tau_{ij}, \psi_{ij})]$
$q \otimes Y = [q \otimes \psi_{ij}] = [q + \psi_{ij}] = [\psi_{ij} + q] = Y \otimes q.$

Also suppose $X = [\tau_{ij}]$ be $m \times n$ matrix and $Y = [\psi_{ij}]$ be $n \times p$ matrix with elements in $(\mathbb{R}_{\max})$, then:

XY is the $m \times p$ matrix whose i, j entry is $(\tau_{i1} \otimes \psi_{1j}) \oplus (\tau_{i2} \otimes \psi_{2j}) \oplus \cdots (\tau_{in} \otimes \psi_{nj}) = max(\tau_{ik} + \psi_{kj})$

3.1 Numerical Examples of Max-plus on Matrix Operations

Let $X = \begin{bmatrix} 3 & 0 \\ -2 & 4 \end{bmatrix}$, $Y = \begin{bmatrix} 6 & 1 \\ -\infty & 9 \end{bmatrix}$ and $q = 2$, where $X, Y \in \mathbb{R}_{\max}^{n \times n}$

3.1.1 Matrix addition $(X \oplus Y)$

$$X \oplus Y = \begin{bmatrix} 3 & 0 \\ -2 & 4 \end{bmatrix} \oplus \begin{bmatrix} 6 & 1 \\ -\infty & 9 \end{bmatrix}$$

$$X \oplus Y = \begin{bmatrix} 3 \oplus 6 & 0 \oplus 1 \\ -2 \oplus -\infty & 4 \oplus 9 \end{bmatrix}$$

$$X \oplus Y = \begin{bmatrix} 6 & 1 \\ -2 & 9 \end{bmatrix}$$

3.1.2 Scalar multiplication $(q \otimes X)$

$$q \otimes X = 2 \otimes \begin{bmatrix} 3 & 0 \\ -2 & 4 \end{bmatrix}$$

$$q \otimes X = \begin{bmatrix} 2 \otimes 3 & 2 \otimes 0 \\ 2 \otimes -2 & 2 \otimes 4 \end{bmatrix}$$

$$q \otimes X = \begin{bmatrix} 5 & 2 \\ 0 & 6 \end{bmatrix}$$

3.1.3 Matrix multiplication $(X \otimes Y)$

$$X \otimes Y = \begin{bmatrix} 3 & 0 \\ -2 & 4 \end{bmatrix} \otimes \begin{bmatrix} 6 & 1 \\ -\infty & 9 \end{bmatrix}$$

$$X \otimes Y = \begin{bmatrix} (3 \otimes 6) \oplus (0 \otimes -\infty) & (3 \otimes 1) \oplus (0 \otimes 9) \\ (-2 \otimes 6) \oplus (4 \otimes -\infty) & (-2 \otimes 1) \oplus (4 \otimes 9) \end{bmatrix}$$

$$X \otimes Y = \begin{bmatrix} 9 \oplus -\infty & 4 \oplus 9 \\ 4 \oplus -\infty & -1 \oplus 13 \end{bmatrix}$$

$$X \otimes Y = \begin{bmatrix} 9 & 9 \\ 4 & 13 \end{bmatrix}$$

Multiplication of matrices in $(\mathbb{R}_{\max}, \oplus, \otimes)$ is associative, that is, $X \otimes (Y \otimes Z) = (X \otimes Y) \otimes Z$ but not commutative , thus, $X \otimes Y \neq$ Y $\otimes$ X. It is only commutative when $X = Y$ or when one of them is a unit matrix. This is where X, Y and Z are matrices with entries from $\mathbb{R}_{max}$.

4 Systems of Equations in Max-algebra

Let $Bx = \psi$, where B is a matrix and ψ and x is a vector of any allowable adimension. $Bx = \psi$ can be rewritten into the following detailed matrix equation and then the equivalent system of max-plus equations:

$$Bx = \psi$$

$$\begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & a_{n2} & \cdots & b_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \vdots \\ \psi_n \end{pmatrix}$$

$$\begin{array}{ccccccccc} (b_{11} \otimes x_1) & \oplus & (b_{12} \otimes x_2) & \oplus & \cdots & \oplus & (b_{1n} \otimes x_n) & = & \psi_1 \\ (b_{21} \otimes x_1) & \oplus & (b_{22} \otimes x_2) & \oplus & \cdots & \oplus & (b_{2n} \otimes x_n) & = & \psi_2 \\ \vdots & \vdots & \vdots & \vdots & & & & & \vdots \\ (b_{n1} \otimes x_1) & \oplus & (b_{n2} \otimes x_2) & \oplus & \cdots & \oplus & (b_{nn} \otimes x_n) & = & \psi_n \end{array}$$

Written in standard notation, the following system is solved simultaneously:

$$\max\{(b_{11} + x_1),(b_{12} + x_2), \cdots, (b_{1n} + x_n)\} = \psi_1$$
$$\max\{(b_{21} + x_1),(b_{22} + x_2), \cdots, (b_{2n} + x_n)\} = \psi_2$$
$$\vdots$$
$$\max\{(b_{n1} + x_1),(b_{n2} + x_2), \cdots, (b_{nn} + x_n)\} = \psi_n$$

First, we consider the case that a solution exists and some of the entries of ψ is $-\infty$. Without loss of generality, the equations can be reordered so that the finite entries of ψ occur first:

$$\begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} \psi_1 \\ \vdots \\ \psi_k \\ -\infty \\ \vdots \\ -\infty \end{pmatrix}$$

Written in standard notation, the following system of equations are obtained:

$$max(b_{11} + x_1, b_{12} + x_2, \cdots, b_{1n} + x_n) = \psi_1$$
$$\vdots$$
$$max(b_{k1} + x_1, b_{k2} + x_2, \cdots, b_{kn} + x_n) = \psi_k$$
$$max(b_{(k+1,1)} + x_1,\ b_{(k+1,2)} + x_2,\ \cdots,\ b_{(k+1,n)} + x_n) = -\infty$$
$$\vdots$$
$$max(b_{n1} + x_1,\ b_{n2} + x_2,\ \cdots,\ b_{nn} + x_n) = -\infty$$

The finite part of B is assumed to be B_1 with dimensions $k \times l$, that of ψ be $\psi' = [\psi_1, \cdots, \psi_k]'$ and that of x be $x' = (x_1, \cdots, x_l)'$

It can be noted that if $Bx = \psi$ has a solution , then $x_{k+1} = x_n = -\infty$, and $Bx' = \psi'$. Thus, $Bx = \psi$ has a solution if and only if x' is a solution to $B_1x' = \psi'$ and solutions to $Bx = \psi$ are

$x = [x', -\infty, \cdots, -\infty]'$

The solvability of a system with infinite entries in ψ can consequently be reduced to that of a system where all the entries in ψ are finite. For that reason attention will be limited to systems $Bx = \psi$ where all the entries of ψ are finite. If there is to be a solution to the system of max-plus equations, then $b_{ij} + x_j \leq \psi$ for all $i \in \{1, ..., n\}$ and $j \in \{1, ..., n\}$. To find a solution to the system, firstly consider each component of x separately. When x_1 is considered for example, if there is a solution to the system, then $b_{i1} + x_1 \leq \psi_i$ for $i = 1, 2, 3, ..., n$. Thus $x_1 \leq \psi_i - b_{i1}$ for each i leads to the following system of upper bounds on x_1:

$$\begin{array}{ccccc} x_1 & \leq & \psi_1 & - & b_{11} \\ x_1 & \leq & \psi_2 & - & b_{21} \\ & \vdots & & & \\ x_1 & \leq & \psi_n & - & b_{n1} \end{array}$$

If this system of inequalities has a solution, then it satisfies:

$$x_1 \leq min\{(\psi_1 - b_{11}), (\psi_2 - b_{21}), ..., (\psi_n - b_{n1})\}$$

Similarly, the possible solutions for $x_2, \cdots, x_n$ can be found, giving the following system of inequalities on the entries of x:

$$\begin{array}{c} x_1 \leq min\{(\psi_1 - b_{11}), (\psi_2 - b_{21}), \cdots, (\psi_n - b_{n1})\} \\ x_2 \leq min\{(\psi_1 - b_{12}), (\psi_2 - b_{22}), \cdots, (\psi_n - b_{n2})\} \\ \vdots \\ x_n \leq min\{(\psi_1 - b_{1n}), (\psi_2 - b_{2n}), \cdots, (\psi_n - b_{nn})\} \end{array}$$

This leads to the candidate for the solution to $Bx = \psi$, which will be denoted by x'.

$$x' = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \quad \text{where} \quad \begin{array}{c} x_1 \leq \min\{(\psi_1 - b_{11}), (\psi_2 - b_{21}), ..., (\psi_n - b_{n1})\} \\ x_2 \leq \min\{(\psi_1 - b_{12}), (\psi_2 - b_{22}), ..., (\psi_n - b_{n2})\} \\ \vdots \\ x_n \leq min\{(\psi_1 - b_{1n}), (\psi_2 - b_{2n}), ..., (\psi_n - b_{nn})\} \end{array}$$

To simplify the process of solving a system of max-plus equations, another matrix can be introduced. The discrepancy matrix, $D_{B.\psi}$ can be define as follows:

$$\begin{pmatrix} \psi_1 - b_{11} & \psi_1 - b_{12} & \cdots & \psi_1 - b_{1n} \\ \psi_2 - b_{21} & \psi_2 - b_{22} & \cdots & \psi_2 - b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \psi_n - b_{n1} & \psi_n - b_{n2} & \cdots & \psi_n - b_{nn} \end{pmatrix}$$

Note that $D_{B.\psi}$ is simply a matrix with all the upper bounds of the x_i's and that each x_i can be found by taking the minimum of the jth column of $D_{B.\psi}$.

Another matrix is formed from $D_{B.\psi}$ called reduced discrepancy matrix, $R_{B.\psi}$:
$R_{B.\psi} = (r_{ij})$ where

$$r_{ij} = \begin{cases} 1 & \text{if } d_{ij}\text{=mininum of column j} \\ 0 & \text{otherwise} \end{cases}$$

$R_{B.\psi}$ is useful in predicting the number of solutions to the matrix equation $Bx = \psi$.

4.1 Solving Systems of Equations in Max-algebra

Example 4.1. *Max-plus system with One solution*

To solve $Bx = \psi$*, where* $B = \begin{bmatrix} 1 & -9 & 4 \\ -4 & 18 & -8 \\ 2 & 1 & -4 \end{bmatrix}$, $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, *and* $\psi = \begin{bmatrix} 1 \\ -6 \\ -3 \end{bmatrix}$

Calculate the discrepancy matrix: $D_{B.\psi} = \begin{bmatrix} 0 & 10 & -3 \\ -2 & -24 & 2 \\ -5 & -4 & 1 \end{bmatrix}$

Taking the minimum of each column of $D_{A.b}$ gives the solution

$x'_1 = min(0, -2, -5) = -5$
$x'_2 = min(10, -24, -4) = -24$
$x'_3 = min(-3, 2, 1) = -3$

The candidate solution to $Bx = \psi$ becomes $x' = (-5, -24, -3)^T$. It can be verified that this is the only solution to $Bx = \psi$ by substituting it back in:

$$\begin{bmatrix} 1 & -9 & 4 \\ -4 & 18 & -8 \\ 2 & 1 & -4 \end{bmatrix} \begin{bmatrix} -5 \\ -24 \\ -3 \end{bmatrix} = \begin{bmatrix} \max(-4, -33, 1) \\ \max(-9, -6, -11) \\ max(-3, -23, -7) \end{bmatrix} = \begin{bmatrix} 1 \\ -6 \\ -3 \end{bmatrix}$$

This will therefore be the only solution to the matrix equation as it will be shown later.

Example 4.2. *Max-plus system with Infinitely many solutions*

To solve $Bx = \psi$*, where* $B = \begin{bmatrix} 1 & 1 & 3 \\ 2 & -1 & 0 \\ 4 & 0 & -1 \end{bmatrix}$, $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, *and* $\psi = \begin{bmatrix} 6 \\ 3 \\ 2 \end{bmatrix}$

Calculate the discrepancy matrix: $D_{B.\psi} = \begin{bmatrix} 5 & 5 & 3 \\ 1 & 4 & 3 \\ -2 & 2 & 3 \end{bmatrix}$

Taking the minimum of each column of $D_{B.\psi}$ gives the solution

$x'_1 = min(5, 1, -2) = -2$
$x'_2 = min(5, 4, 2) = 2$
$x'_3 = min(3, 3, 3) = 3$

The candidate solution to $Bx = \psi$ becomes $x' = (-2, 2, 3)^T$. This solution can be verified by substituting it back in

$$\begin{bmatrix} 1 & 1 & 3 \\ 2 & -1 & 0 \\ 4 & 0 & -1 \end{bmatrix} \begin{bmatrix} -2 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} max(-1, 3, 6) \\ max(0, 1, 3) \\ max(2, 2, 2) \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \\ 2 \end{bmatrix}$$

X' is therefore a solution to the given matrix equation. It can be seen that there are other feasible solution. Any x of the form $\{x : x = (u, v, 3)^T$ where $u \leq -2$ and $v \leq 2\}$ is also a solution to the given matrix equation.

Example 4.3. *Max-plus system with No solutions*

To solve $Bx = \psi$*, where* $B = \begin{bmatrix} 2 & -1 & -1 \\ 0 & 4 & 3 \\ 1 & 2 & 0 \end{bmatrix}$, $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, *and* $b = \begin{bmatrix} 5 \\ 5 \\ 7 \end{bmatrix}$

The discrepancy matrix: $D_{B.\psi} = \begin{bmatrix} 3 & 6 & 6 \\ 5 & 1 & 2 \\ 6 & 5 & 7 \end{bmatrix}$
Which gives the solution of $x' = (3, 1, 2)^T$.
x' is verified to see that it is not a solution.

$$\begin{bmatrix} 2 & -1 & -1 \\ 0 & 4 & 3 \\ 1 & 2 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} max(5,0,1) \\ max(3,5,5) \\ max(4,3,2) \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \\ \underline{4} \end{bmatrix} \neq b = \begin{bmatrix} 5 \\ 5 \\ 7 \end{bmatrix}$$

It can be noticed that the underlined entry is not in congruous with the entry of ψ. However, a solution x must satisfy $x_1 \leq 3$, $x_2 \leq 1$, and $x_3 \leq 2$ because the components of x' are the upper bounds. It can be seen from the third row that $max(x_1 + 1, x_2 + 2, x_3 + 0) \leq 4 < 7$.

A reduced discrepancy matrix $R_{B.\psi}$ is use to predict the number of solutions to the matrix equation $Bx = \psi$. The table below shows the various examples and their $D_{B.\psi}$ and $R_{B.\psi}$. Where the minimum occurs in each column of $D_{B.\psi}$ has been underlined for each entries. Note that they are the 'one' entries of each correspond $R_{B.\psi}$.

Table 1. Example of the Various solutions and their minimum entries underligned

Example	$D_{B.\psi}$	$R_{B.\psi}$
One Solution	$\begin{bmatrix} 0 & 10 & \underline{-3} \\ -2 & \underline{-24} & 2 \\ \underline{-5} & -4 & 1 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & \underline{1} \\ 0 & \underline{1} & 0 \\ \underline{1} & 0 & 0 \end{bmatrix}$
Infinite Solutions	$\begin{bmatrix} 5 & 5 & \underline{3} \\ 1 & 44 & \underline{3} \\ \underline{-2} & \underline{2} & \underline{3} \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & \underline{1} \\ 0 & 0 & \underline{1} \\ \underline{1} & \underline{1} & \underline{1} \end{bmatrix}$
No Solutions	$\begin{bmatrix} \underline{3} & 6 & 6 \\ 5 & \underline{1} & \underline{2} \\ 6 & 5 & 7 \end{bmatrix}$	$\begin{bmatrix} \underline{1} & 0 & 0 \\ 0 & \underline{1} & \underline{1} \\ 0 & 0 & 0 \end{bmatrix}$

The minimum entry of the column, j, in the $D_{B.\psi}$ matrix is the maximum solution to the system of inequalities for x_j. To alter this system of inequalities to a system of equalities, there must be an equality in each row inequality, thus, there must be at least one minimum in each row of $D_{B.\psi}$ there must be at least one in each row of $R_{B.\psi}$ for a solution to exist.

'1' in the *jth* column of $R_{B.\psi}$ signifies the minimum of the upper bounds for x_j. If there are no other '1s' in the row where a '1' occurs, the only way that the equation corresponding to that row can be solved is to have x_j achieve the bound. This causes the value of x_j to be fixed at a specific value, making it a variable-fixing entry. This can be illustrated by underlining the variable-fixing entries for the examples in the table below:

Table 2. Variable-fixing entries

Example	$R_{B.\psi}$
One Solution	$\begin{bmatrix} 0 & 0 & \underline{1} \\ 0 & \underline{1} & 0 \\ \underline{1} & 0 & 0 \end{bmatrix}$
Infinite Solutions	$\begin{bmatrix} 0 & 0 & \underline{1} \\ 0 & 0 & \underline{1} \\ 1 & 1 & \underline{1} \end{bmatrix}$
No Solutions	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

From the table above, to consider the $R_{B.\psi}$ for one Solution, all the non-zero entries are variable-fixing entries. The first row equation fixes the x_3 component where $x_3 = -3$. The second row equation fixes the x_2 component where $x_2 = -24$. Finally, the third row equation fixes the x_1 component where $x_1 = -5$ making all the components of x to be fixed.

There are slack entries in $R_{B.\psi}$ for Infinite Solutions. The first row equation fixes the x_3 component, $x_3 = 3$. The component solution to the second row equation has already been fixed by the first row equation. In the third row equation, there are three possible ways for equality to be achieved, is either $x_1 = -2$, $x_2 = 2$ or $x_3 = 3$. But x_3 which is 3, has already been fixed. As long as $x_1 \leq -2$ and $x_2 \leq 2$, no problem can be caused.

For $R_{B.\psi}$ in the example of No Solution, because there exist a third row of $R_{B.\psi}$ containing zeros (or no 1's), there is No solution for the system of equations which does not fulfil the condition that there must be at least one minimum in each row of $D_{B.\psi}$, thus there must be at least a '1' in each row of $R_{B.\psi}$ for a solution to exist.

The above analysis explain that the method used works for all $n \times n$ system of equations.

Example 4.4. *Max-plus system with only One solution*

To solve $Bx = \psi$, *where* $B = \begin{bmatrix} 5 & -1 & -1 & -1 \\ -1 & 4 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ -1 & -1 & -1 & 4 \end{bmatrix}$, $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$, *and* $\psi = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$

$$\text{The discrepancy matrix: } D_{B.\psi} = \begin{bmatrix} -4 & 2 & 2 & 2 \\ 2 & -3 & 2 & 2 \\ 2 & 2 & -2 & 2 \\ 2 & 2 & 2 & -3 \end{bmatrix}$$

Taking the minimum of each column of $D_{B.\psi}$ gives the solution

$x_1' = min(-4, 2, 2, 2) = -4$
$x_2' = min(2, -3, 2, 2) = -3$
$x_3' = min(2, 2, -2, 2) = -2$

$x'_4 = min(2, 2, 2, -3) = -3$

The candidate solution to $Bx = \psi$ becomes $x' = (-4, -3, -2, -3)^T$.

This solution can be verified by substituting it back in

$$\begin{bmatrix} 5 & -1 & -1 & -1 \\ -1 & 4 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ -1 & -1 & -1 & 4 \end{bmatrix} \begin{bmatrix} -4 \\ -3 \\ -2 \\ -3 \end{bmatrix} = \begin{bmatrix} max(1, -4, -3, -4) \\ max(-5, 1, -3, -4) \\ max(-5, -4, 1, -4) \\ max(-5, -4, -3, 1) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

This will be the only solution to the matrix equation which we will show later by the reduced discrepancy ($R_{B.\psi}$).

Example 4.5. *Max-plus system with Infinitely many solutions*

To solve $Bx = \psi$, *where* $B = \begin{bmatrix} 4 & -1 & 1 & 1 \\ -1 & 3 & -1 & -1 \\ -1 & 0 & -3 & -1 \\ 1 & 1 & 0 & -2 \end{bmatrix}$, $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$, *and* $\psi = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}$

Calculate the discrepancy matrix: $D_{A.b} = \begin{bmatrix} -3 & 2 & 0 & 0 \\ 1 & -3 & 1 & 1 \\ 0 & -1 & 2 & 0 \\ -1 & -1 & 0 & 2 \end{bmatrix}$

which gives the solution of $x' = (-3, -3, 0, 0)^T$. This solution can be verified by substituting it back in

$$\begin{bmatrix} 4 & -1 & 1 & 1 \\ -1 & 3 & -1 & -1 \\ -1 & 0 & -3 & -1 \\ 1 & 1 & 0 & -2 \end{bmatrix} \begin{bmatrix} -3 \\ -3 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} max(1, -4, -2, -2) \\ max(-4, 0, -4, -4) \\ max(-4, -3, -3, -1) \\ max(-2, -2, 0, -2) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}$$

x' is therefore a solution to the given matrix equation. There are other solutions that also work. Any x of the form $\{x : x = (u, -3, 0, 0)^T$, where $u \leq -3\}$ is also a solution to the given matrix equation.

Example 4.6. *Max-plus system with No solutions*

To solve $Bx = \psi$, *where* $B = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 4 \\ -1 & 1 & -1 & -1 \\ -1 & 3 & 1 & -1 \end{bmatrix}$, $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$, *and* $b = \begin{bmatrix} 2 \\ 1 \\ -6 \\ -2 \end{bmatrix}$

The discrepancy matrix: $D_{B.\psi} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -1 & -1 & -3 \\ -5 & -7 & -5 & -5 \\ -1 & -5 & -3 & -1 \end{bmatrix}$ which gives the solution of $x' = (-5, -7, -5, -5)^T$.

x' is verified to see that it is not a solution

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 4 \\ -1 & 1 & -1 & -1 \\ -1 & 3 & 1 & -1 \end{bmatrix} \begin{bmatrix} -5 \\ -7 \\ -5 \\ -5 \end{bmatrix} = \begin{bmatrix} max(-4, -6, -4, -4) \\ max(-4, -5, -3, -1) \\ max(-6, -6, -6, -6) \\ max(-6, -4, -4, -6) \end{bmatrix} = \begin{bmatrix} \underline{-1} \\ \underline{-1} \\ -6 \\ \underline{-4} \end{bmatrix} \neq b = \begin{bmatrix} 2 \\ 1 \\ -6 \\ -2 \end{bmatrix}$$

It is noticed that the underlined entries congruos with the entry of ψ. A solution X must satisfy $x_1 \leq -5$, $x_2 \leq -7$, $x_3 \leq -5$ and $x_4 \leq -5$ because the components of X' are the upper bounds.

It is seen from the first, second and fourth row that $max(x_1 + 1, x_2 + 1, x_3 + 1, x_4 + 1) \leq -1 < 2$, $max(x_1 + 1, x_2 + 2, x_3 + 2, x_4 + 4) \leq -1 < 1$, and $max(x_1 - 1, x_2 + 3, x_3 + 1, x_4 - 1) \leq -4 < -2$ respectively. The matrix equation therefore has No solution.

The table below shows the various examples and their $D_{B.\psi}$ and $R_{B.\psi}$.

Table 3. Max-plus system with One, Infinite and No solutions, and their corresponding $D_{B.\psi}$ and $R_{B.\psi}$.

Example	$D_{B.\psi}$	$R_{B.\psi}$
One Solution	$\begin{bmatrix} \underline{-4} & 2 & 2 & 2 \\ 2 & \underline{-3} & 2 & 2 \\ 2 & 2 & \underline{-2} & 2 \\ 2 & 2 & 2 & \underline{-3} \end{bmatrix}$	$\begin{bmatrix} \underline{1} & 0 & 0 & 0 \\ 0 & \underline{1} & 0 & 0 \\ 0 & 0 & \underline{1} & 0 \\ 0 & 0 & 0 & \underline{1} \end{bmatrix}$
Infinite Solutions	$\begin{bmatrix} \underline{-3} & 2 & \underline{0} & \underline{0} \\ 1 & \underline{-3} & 1 & 1 \\ 0 & -1 & 2 & \underline{0} \\ -1 & -1 & \underline{0} & 2 \end{bmatrix}$	$\begin{bmatrix} \underline{1} & 0 & \underline{1} & \underline{1} \\ 0 & \underline{1} & 0 & 0 \\ 0 & 0 & 0 & \underline{1} \\ 0 & 0 & \underline{1} & 0 \end{bmatrix}$
No Solutions	$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -1 & -1 & -3 \\ \underline{-5} & \underline{-7} & \underline{-5} & \underline{-5} \\ -1 & -5 & -3 & -1 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \underline{1} & \underline{1} & \underline{1} & \underline{1} \\ 0 & 0 & 0 & 0 \end{bmatrix}$

To consider $R_{B.\psi}$ for One Solution, all the non-zero entries are variable-fixing entries. The first row equation fixes the x_1 component where $x_1 = -4$. The second row equation also fixes x_2, where $x_2 = -3$. The third row also fixes x_3, where $x_3 = -2$. Finally, the fourth row equation fixes x_4, where $x_4 = -3$. This has made all the x components to be fixed.

There are slack entries in $R_{B.\psi}$ for Infinite solutions. The first row has three feasible solutions to achieve equality, is either $x_1 = -3$, $x_3 = 0$ or $x_4 = 0$. x_3 component is chosen, where $x_3 = 0$. The second row fixes x_2, where $x_2 = -3$. The third row equation fixes the x_4 component, where $x_4 = 0$. The component solution to the fourth row equation has already been fixed by the first row.

For the $R_{B.\psi}$ in No solutions, there are three rows, thus, the first, second and fourth containing zeros (no 1's). This does not fulfil the condition for a solution to exist. Therefore the system of equations has No solutions.

We also applied this discrepancy method to a system of $m \times n$ equations where $m < n$. An example of such a system that we used was a 3-by-4 systems of equations.

Example 4.7. *Max-plus system with Infinite solutions*

To solve $Bx = \psi$, *where* $B = \begin{bmatrix} 0 & -1 & 1 & 1 \\ 1 & 2 & 2 & -1 \\ 1 & 3 & 1 & 0 \end{bmatrix}$, $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$, *and* $\psi = \begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix}$

The discrepancy matrix: $D_{B.\psi} = \begin{bmatrix} 2 & 3 & 1 & 1 \\ 2 & 1 & 1 & 4 \\ 1 & -1 & 1 & 2 \end{bmatrix}$ which gives the solution of $x' = (1, -1, 1, 1)^T$.
X' is therefore a solution to the given matrix equation.

$$\begin{bmatrix} 0 & -1 & 1 & 1 \\ 1 & 2 & 2 & -1 \\ 1 & 3 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} max(1,-2,2,2) \\ max(2,1,3,0) \\ max(2,2,2,1) \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix}$$

It can be seen that there are other solutions that is also feasible. Any x of the form $\{x : x = (u, v, 1, w)^T$, where $u \leq 1$, $v \leq -1$, and $w \leq 1\}$ is also a solution to the given matrix equation.

Example 4.8. *Max-plus system with No solutions*

To solve $Bx = \psi$, *where* $B = \begin{bmatrix} -3 & 2 & 1 & 4 \\ 0 & 5 & 3 & -1 \\ -1 & 6 & 0 & 8 \end{bmatrix}$, $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$, *and* $\psi = \begin{bmatrix} 10 \\ 14 \\ 9 \end{bmatrix}$

The discrepancy matrix: $D_{B.\psi} = \begin{bmatrix} 13 & 8 & 9 & 6 \\ 14 & 9 & 11 & 15 \\ 10 & 3 & 9 & 1 \end{bmatrix}$ which gives the solution of $x' = (10, 3, 9, 1)^T$.
x' is verified to see that it is not a solution

$$\begin{bmatrix} -3 & 2 & 1 & 4 \\ 0 & 5 & 3 & -1 \\ -1 & 6 & 0 & 8 \end{bmatrix} \begin{bmatrix} 10 \\ 3 \\ 9 \\ 1 \end{bmatrix} = \begin{bmatrix} max(7,5,10,5) \\ max(10,8,12,0) \\ max(9,9,9,9) \end{bmatrix} = \begin{bmatrix} 10 \\ \underline{12} \\ 9 \end{bmatrix} \neq b = \begin{bmatrix} 10 \\ 14 \\ 9 \end{bmatrix}$$

The underlined entry does not correspond the entry of ψ. A solution x must satisfy $x_1 \leq 10$, $x_2 \leq 3$, $x_3 \leq 9$, and $x_4 \leq 1$ since the components of x' are the upper bounds. From the second row $max(x_1 + 0, x_2 + 5, x_3 + 3, x_4 - 1) \leq 12 < 14$. This makes the matrix equation to have no solution. The table below shows the various examples and their $D_{B.\psi}$ and $R_{B.\psi}$.

Table 4. Max-plus system with Infinite and No solutions

Example	$D_{B.\psi}$	$R_{B.\psi}$
Infinite Solutions	$\begin{bmatrix} 2 & 3 & \underline{1} & \underline{1} \\ 2 & 1 & \underline{1} & 4 \\ \underline{1} & \underline{-1} & \underline{1} & 2 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & \underline{1} & \underline{1} \\ 0 & 0 & \underline{1} & 0 \\ \underline{1} & \underline{1} & \underline{1} & 0 \end{bmatrix}$
No Solutions	$\begin{bmatrix} 13 & 8 & \underline{9} & 6 \\ 14 & 9 & 11 & 15 \\ \underline{10} & \underline{3} & \underline{9} & \underline{1} \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & \underline{1} & 0 \\ 0 & 0 & 0 & 0 \\ \underline{1} & \underline{1} & \underline{1} & \underline{1} \end{bmatrix}$

From the $R_{B.\psi}$ in the Infinite Solutions, there are slack entries. In the first row equation, there are two possible ways for equality to be attained, is either $x_3 = 1$ or $x_4 = 1$. The x_3 component is fixed, where $x_3 = 1$ for the first row. The second row has already been fixed by the first row. The third row also has either $x_1 = 1$, $x_2 = -1$ or $x_3 = 1$ for equality to be achieved. But x_3 which is 1 has already been fixed. To consider $R_{B.\psi}$ in No solutions, there is the second row which is having all zeros (no 1's). Therefore the system of equations has no solution.

4.2 Example on a System of Max-linear Program

Example 4.9. *Given a system of max-linear program in which* $f = (9, 5, 2, 7)^T$ *for*

$$B \otimes x = \psi$$
$$C \otimes x \leq \gamma$$

where

$$B = \begin{pmatrix} 0 & -1 & 1 & 1 \\ 1 & 2 & 2 & -1 \\ 1 & 3 & 1 & 0 \end{pmatrix}, \psi = \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix}$$
$$C = \begin{pmatrix} -3 & 2 & 1 & 4 \\ 0 & 5 & 3 & -1 \\ -1 & 6 & 0 & 8 \end{pmatrix}, \gamma = \begin{pmatrix} 10 \\ 14 \\ 9 \end{pmatrix}$$

This is the solution:

$$D_{B.\psi} = \begin{pmatrix} 2 & 3 & 1 & 1 \\ 2 & 1 & 1 & 4 \\ 1 & -1 & 1 & 2 \end{pmatrix}$$

$$\bar{x}(B, \psi) = \begin{pmatrix} min & (2, 2, 1) \\ min & (3, 1, -1) \\ min & (1, 1, 1) \\ min & (1, 4, 2) \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 1 \\ 1 \end{pmatrix}$$

$$D_{C.\gamma} = \begin{pmatrix} 13 & 8 & 9 & 6 \\ 14 & 9 & 11 & 15 \\ 10 & 3 & 9 & 1 \end{pmatrix}$$

$$\bar{x}(C, \gamma) = \begin{pmatrix} min & (13, 14, 10) \\ min & (8, 9, 3) \\ min & (9, 11, 9) \\ min & (6, 15, 1) \end{pmatrix} = \begin{pmatrix} 10 \\ 3 \\ 9 \\ 1 \end{pmatrix}$$

Compare $\bar{x}(A, b)$ and $\bar{x}(C, d)$, and pick the least corresponding elements to form $\hat{x}(B, C, \psi, \gamma)$

$$\hat{x}(B, C, \psi, \gamma) = \begin{pmatrix} 1 \\ -1 \\ 1 \\ 1 \end{pmatrix}$$

We denote

$$x = \hat{x}(B, C, \psi, \gamma) = \begin{pmatrix} 1 \\ -1 \\ 1 \\ 1 \end{pmatrix}$$

Compare the corresponding elements of $\bar{x}(B, \psi)$ and $\bar{x}(C, \gamma)$ that satisfy $\bar{x}_j(C, d) \geq \bar{x}_j(B, \psi)$ and pick their positions (in the row), making $J = \{1, 2, 3, 4\}$

From K_j, where $j \in J$

$K_1 = \{3\}$, $K_2 = \{3\}$, $K_3 = \{1, 2, 3\}$, and $K_4 = \{1\}$

$$\begin{aligned} f(x) &= f_j + x_j \\ &= ((9 + 1), (5 + -1), (2 + 1), (7 + 1)) \\ &= (10, 4, 3, 8)^T \end{aligned}$$

$$H(x) = \{1\}$$
$$J : J \setminus H(x) = \{2, 3, 4\}$$
$$K = \{1, 2, 3\}$$
$$K_2 \cup K_3 \cup K_4 = \{1, 2, 3\} = K$$

set $x_1 = 10^{-5}$ (say), we get a new $x = (10^{-5}, -1, 1, 1)^T$

Going for a new $H(x)$

$$\begin{aligned} f(x) &= f_j + x_j \\ &= ((9 + 10^{-5}), (5 + -1), (2 + 1), (7 + 1)) \\ &= (9.00001, 4, 3, 8)^T \end{aligned}$$

$$H(x) = \{4\}$$
$$J : J \setminus H(x) = \{2, 3\}$$
$$K_2 \cup K_3 = \{1, 2, 3\} = K$$

set $x_4 = 10^{-5}$ (say), we get a new $x = (10^{-5}, -1, 1, 10^{-5})^T$

Going for a new $H(x)$

$$\begin{aligned} f(x) &= f_j + x_j \\ &= ((9 + 10^{-5}), (5 + 10^{-5}), (2 + 1), (7 + 10^{-5})) \\ &= (9.00001, 5.00001, 3, 7.00001)^T \end{aligned}$$

$$H(x) = \{3\}$$
$$J : J \setminus H(x) = \{2\}$$
$$K_2 \neq K$$

We stop, the optimal solution is $x = (10^{-5}, 10^{-5}, 1, 10^{-5})^T$
$f^{min} = minf(x) = min(9.00001, 5.00001, 3, 7.00001)^T$
Therefore $f^{min} = 3$

5 Conclusion

From the examples illustrated in this paper, it is clear that for an $n \times n$ system of linear equations in $(\mathbb{R}_{max}, \oplus, \otimes)$, we had either One solution, an Infinite number of solutions or No solution. The same applies to an $n \times n$ system of linear equations in $(\mathbb{R}, +, \cdot)$, where either One solution, an Infinite number of solutions or No solution can be formed.

It is also interesting to note that an $m \times n$ system of linear equations (where $m < n$) has either an Infinite number of solutions or No solution in $(\mathbb{R}_{max}, \oplus, \otimes)$. The same applies to an $m \times n$ system of linear equations in $(\mathbb{R}, +, \cdot)$ where $m < n$. Linear programming problem involving a linear equations and inequalities has also been solved using max-plus.

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Competing Interests

The authors declare that no competing interests exist.

References

[1] Cuninghame-Green RA. Minimax algebra. Lecture notes in Economics and Mathematical Systems. 1979;166.

[2] Gaubert S. Methods and applications of (max,+) linear algebra. INRIA Rocquincourt; 1997.

[3] Gaubert S. The minkowski theorem for max-plus convex sets. Linear Algebra and its Applications. 2007;421:356-369.

[4] Tomàskovà H. Circulant matrices in extremal algebras. Abstracts of the Conf. CJS 2011, Hejnice; 2011.

[5] Peter B. Strong regularity of matrices - a survey of results. Discrete Applied Mathematics. 1994;48:45-68.

[6] Olsder G, Baccelli F, Cohen G, Quadrat J. Course notes: Max-algebra approach to discrete event systems. Algebres Max-plus et Applications an Informatique et Automatique, Inria. 1998;147-196.

[7] Andersen H. Max-plus algebra: Properties and Applications; 2002.

[8] Hungerford T. Algebra: Graduate tests in mathematics. Springer-Verlag, New York; 1974.

[9] Aminu A. Max-algebraic linear systems and programs. PhD Thesis, University of Birmingham; 2009.

[10] Aminu A. Simultaneous solution of linear equations and inequalities in max - algebra. Kybernetika. 2011;49:241-250.

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