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Generalized Hyers-Ulam Stability of a Mixed Type Additive-quadratic Functional Equation in non-Archimedean Fields

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Abstract

In this paper, we investigate the generalized Hyers-Ulam stability of the functional equation

$$4[f(x+3y)+f(3x+y)]-9f(x+y)+15f(x-y) \\ = 4f(3x)+10f(x)+9f(3y)-35f(y)$$

in non-Archimedean fields.

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1 Introduction

In 1940, S. M. Ulam [1] raised the following question concerning the stability of group homomorphisms:

"Let G be a group and H be a metric group with metric $d(.,.)$. Given $\epsilon > 0$, does there exist a $\delta > 0$ such that if a function $f : G \to H$ satisfies

$$d(f(xy), f(x)f(y)) < \delta$$

for all $x, y \in G$, then there exists a homomorphism $a : G \to H$ with $d(f(x), a(x)) < \epsilon$ for all $x \in G$?"

In 1941, D. H. Hyers [2] gave an answer to the Ulam's stability problem. In 1950, T. Aoki [3] generalized the Hyers theorem for additive mappings. In 1978, Th.M. Rassias [4] provided a generalized version of the theorem of Hyers which permitted the cauchy difference to become unbounded. The stability phenomenon that was presented by Th.M. Rassias is called the generalized Hyers-Ulam stability.

The functional equation

$$f(x + y) + f(x - y) = 2f(x) + 2f(y)$$

is called a quadratic functional equation. In particular, every solution of the quadratic functional equation is said to be a quadratic mapping. M.S. Moslehian and Th.M. Rassias [5] proved the Hyers-Ulam-Rassias stability of the Cauchy functional equation and the quadratic functional equation in non-Archimedean spaces.

The stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem ([6], [7], [8], [9], [10], [11], [12], [13], [14], [15], [16], [17], [18], [19], [20], [21], [22], [23], [24]).

In this paper, we consider the following functional equation deriving from additive and quadratic functions

$$\begin{aligned} 4[f(x + 3y) + f(3x + y)] - 9f(x + y) + 15f(x - y) \\ = 4f(3x) + 10f(x) + 9f(3y) - 35f(y). \end{aligned} \tag{1.1}$$

It is easy to see that the function $f(x) = ax + bx^2$ is a solution of the functional equation (1.1). In this paper, we obtain the general solution and the generalized Hyers-Ulam stability of the functional equation (1.1) in non-Archimedean fields.

2 Preliminaries

By a *non-Archimedean field* we mean a field K equipped with a function (valuation) $|.|$ from K into $[0, \infty)$ such that $|r| = 0$ if and only if $r = 0$, $|rs| = |r||s|$ and $|r + s| \leq max\{|r|, |s|\}$ for all $r, s \in K$. Clearly $|1| = | - 1| = 1$ and $|n| \leq 1$ for all $n \in \mathbb{N}$.

Let X be a vector space over a scalar field $\mathbb{K}$ with a non-Archimedean non-trivial valuation $|.|$. A function $||.|| : X \to \mathbb{R}$ is a *non-Archimedean norm (valuation)* if it satisfies the following conditions:

(i) $||x|| = 0$ if and only if $x = 0$;

(ii) $||rx|| = |r|||x|| \qquad (r \in \mathbb{K}, x \in X)$;

(iii) the strong triangle inequality (ultrametric); namely,

$$||x + y|| \leq \max\{||x||, ||y||\} \qquad (x, y \in X).$$

Then $(X, ||.||)$ is called a non-Archimedean space. Due to the fact that

$$\|x_n - x_m\| \leq \max\{\|x_{j+1} - x_j\| : m \leq j \leq n - 1\} \qquad (n > m)$$

a sequence $\{x_n\}$ is Cauchy if and only if $\{x_{n+1} - x_n\}$ converges to zero in a non-Archimedean space. By a complete non-Archimedean space we mean one in which every Cauchy sequence is convergent.

An example of a non-Archimedean valuation is the mapping $|.|$ taking everything but 0 into 1 and $|0| = 0$. This valuation is called trivial. Another example of a non-Archimedean valuation on a field $\mathbb{K}$ is the mapping

$$\|x\| = \begin{cases} 0 & \text{if } x = 0 \\ \frac{1}{x} & \text{if } x > 0 \\ -\frac{1}{x} & \text{if } x < 0 \end{cases}$$

for any $x \in \mathbb{K}$.

3 General Solution of the Functional Equation (1.1)

In this section, we obtain the general solution of functional equation (1.1).

Theorem 3.1. Let X, Y be vector spaces. An even function $f : X \to Y$ satisfies the functional equation (1.1) if and only if f is quadratic.

Proof. Let f be an even function. Letting (x, y) by $(0, 0)$ in the functional equation (1.1), we get $f(0) = 0$. Putting $x = 0$ in the functional equation (1.1), we find that $f(3y) = 9f(y)$, for all $y \in X$. Now, setting (x, y) as $(x - y, x + y)$ in the functional equation (1.1), we have

$$8[f(2x + y) + f(2x - y)] - 18f(x) + 30f(y) = 23[f(x + y) + f(x - y)] \tag{3.1}$$

for all $x, y \in X$. Replacing (x, y) by $(x, 0)$ in (3.1), we obtain $f(2x) = 4f(x)$, for all $x \in X$. Switching (x, y) to $(x, 2y)$ in (3.1) and then multiplying by 8, we get

$$8f(x + 2y) + 8f(x - 2y) = \frac{256}{23}[f(x + y) + f(x - y)] - \frac{144}{23}f(x) + \frac{960}{23}f(y) \tag{3.2}$$

for all $x, y \in X$. Interchanging x with y in (3.2), we have

$$8[f(2x + y) + f(2x - y)] = \frac{256}{23}[f(x + y) + f(x - y)] - \frac{144}{23}f(y) + \frac{960}{23}f(x) \tag{3.3}$$

for all $x, y \in X$. Using (3.1) in (3.3), multiplying by $\frac{23}{273}$ and further simplification gives,

$$f(x + y) + f(x - y) = 2f(x) + 2f(y) \tag{3.4}$$

for all $x, y \in X$, which implies that f is quadratic function.

Conversely, suppose that f is quadratic function. Then f satisfies (3.4). Replacing (x, y) by $(3x, x)$ in (3.4) and simplifying further, we get $f(3x) = 9f(x)$, for all $x \in X$. Switching (x, y) to $(2x, x + y)$ in (3.4), we obtain

$$f(3x + y) + f(x - y) = 8f(x) + 2f(x + y) \tag{3.5}$$

for all $x, y \in X$. Interchanging x with y in (3.5), we get

$$f(x + 3y) + f(x - y) = 8f(y) + 2f(x + y) \tag{3.6}$$

for all $x, y \in X$. Now, adding (3.5) with (3.6) and then multiplying the resulting equation by 4, we obtain

$$4[f(3x + y) + f(x + 3y)] = 32f(y) + 32f(x) + 16f(x + y) - 8f(x - y) \tag{3.7}$$

for all $x, y \in X$. Multiplying equation (3.4) by 7 gives,

$$7f(x + y) + 7f(x - y) = 14f(x) + 14f(y) \tag{3.8}$$

for all $x, y \in X$. Summing (3.7) with (3.8) and further simplification yields the functional equation (1.1), which completes the proof. □

Theorem 3.2. Let X, Y be vector spaces. An odd function $f : X \to Y$ satisfies the functional equation (1.1) if and only if f is additive.

Proof. Let f be an odd function. Replacing (x, y) by $(0, y)$ in the functional equation (1.1) and further simplification yields $f(3y) = 3f(y)$, for all $y \in X$. Substituting $(x, y) = (x - y, x + y)$ in the functional equation (1.1) and then dividing by 2, we get

$$4[f(2x + y) + f(2x - y)] - 9f(x) - 15f(y) = 11f(x - y) - 4f(x + y) \tag{3.9}$$

for all $x, y \in X$. Switching (x, y) to $\left(\frac{x}{2}, 0\right)$ in (3.9) and simplifying further, we obtain $f\left(\frac{x}{2}\right) = \frac{1}{2}f(x)$, for all $x \in X$. Putting $y = 0$ in (3.9) and on further simplification, we have $f(2x) = 2f(x)$, for all $x \in X$. Replacing (x, y) by $\left(\frac{x}{2}, y\right)$ in (3.9) and multiplying by 2, we get

$$8[f(x + y) + f(x - y)] - 9f(x) - 30f(y) = 11f(x - 2y) - 4f(x + 2y) \tag{3.10}$$

for all $x, y \in X$. Interchaning x with y in (3.10), we obtain

$$8[f(x + y) - f(x - y)] - 9f(y) - 30f(x) = -11f(2x - y) - 4f(2x + y) \tag{3.11}$$

for all $x, y \in X$. Now, replacing y by $-y$ in (3.11), we get

$$8[f(x - y) - f(x + y)] + 9f(y) - 30f(x) = -11f(2x + y) - 4f(2x - y) \tag{3.12}$$

for all $x, y \in X$. Adding (3.11) with (3.12) and further simplification, yields

$$4f(x) = f(2x + y) + f(2x - y) \tag{3.13}$$

for all $x, y \in X$. Now, replacing (x, y) by $\left(\frac{x}{2}, y\right)$ in (3.13), we obtain

$$2f(x) = f(x + y) + f(x - y) \tag{3.14}$$

for all $x, y \in X$. Interchaning x with y in (3.14), we get

$$2f(y) = f(x + y) - f(x - y) \tag{3.15}$$

for all $x, y \in X$. Adding (3.14) and (3.15), we obtain

$$f(x + y) = f(x) + f(y) \tag{3.16}$$

for all $x, y \in X$, which shows that f is additive.

Conversely, suppose f satisfies (3.16). Replacing x by $3x$ in (3.16) gives

$$f(3x + y) = f(3x) + f(y) \tag{3.17}$$

for all $x, y \in X$. Now, replacing y by $3y$ in (3.16), we get

$$f(x + 3y) = f(x) + f(3y) \tag{3.18}$$

for all $x, y \in X$. Summing (3.17) with (3.18) and then multiplying the resulting equation by 4, we obtain

$$4[f(3x + y) + f(x + 3y)] = 4f(3x) + 4f(3y) + 4f(x) + 4f(y) \tag{3.19}$$

for all $x, y \in X$. Putting $y = 3y$ in (3.16) and simplifying further, we get

$$15f(y) = 5f(3y) \tag{3.20}$$

for all $y \in X$. Multiplying (3.16) by -9, we obtain

$$-9f(x + y) = -9f(x) - 9f(y) \tag{3.21}$$

for all $x, y \in X$. Multiplying (3.16) by 15, we get

$$15f(x - y) = 15f(x) - 15f(y) \tag{3.22}$$

for all $x, y \in X$. Now, summing (3.19), (3.20), (3.21) and (3.22), we arrive at the functional equation (1.1), which completes the proof of theorem. □

Theorem 3.3. Let X, Y be vector spaces, and let $f : X \to Y$ be a function. Then f satisfies the functional equation (1.1) if and only if there exists a unique additive function $A : X \to Y$ and a unique quadratic function $Q : X \to Y$ such that $f(x) = A(x) + Q(x)$ for all $x \in X$.

Proof. Define the mappings $A, Q : X \to Y$ by

$$A(x) = \frac{f(x) - f(-x)}{2} \tag{3.23}$$

and

$$Q(x) = \frac{f(x) + f(-x)}{2} \tag{3.24}$$

for all $x \in X$. Substituting x by $-x$ in (3.23) and (3.24), we get respectively

$$A(-x) = -A(x) \qquad \text{and} \qquad Q(-x) = Q(x) \tag{3.25}$$

for all $x \in X$. Using (3.23) and (3.24) in the functional equation (1.1), we get

$$\begin{aligned} &4A(x + 3y) + 4A(3x + y) - 9A(x + y) + 15A(x - y) \\ &\qquad = 4A(3x) + 9A(3y) + 10A(x) - 35A(y) \end{aligned} \tag{3.26}$$

and

$$\begin{aligned} &4Q(x + 3y) + 4Q(3x + y) - 9Q(x + y) + 15Q(x - y) \\ &\qquad = 4Q(3x) + 9Q(3y) + 10Q(x) - 35Q(y) \end{aligned} \tag{3.27}$$

for all $x, y \in X$.

First, we claim that A is additive. Putting $x = y = 0$ in (3.26), we get $A(0) = 0$. Putting $y = x$ in (3.26) and using (3.25), we get

$$A(3x) = 3A(x) \tag{3.28}$$

for all $x \in X$. Using (3.28) in (3.26), we get

$$4A(x + 3y) + 4A(3x + y) - 9A(x + y) + 15A(x - y) = 22A(x) - 8A(y)$$

for all $x, y \in X$. By Theorem 3.2, A is addiitive. Next, we claim that Q is quadratic. Putting $x = y = 0$ in (3.27), we get $Q(0) = 0$. Putting $y = x$ and using (3.25), we get

$$Q(3x) = 3Q(x) \tag{3.29}$$

for all $x \in X$. Using (3.29) and (3.27), we obtain

$$4Q(3x + y) + 4Q(x + 3y) - 9Q(x + y) + 15Q(x - y) = 46Q(x) + 46Q(y)$$

for all $x, y \in X$. By Theorem 3.1, Q is quadratic. Therefore, we have $f(x) = A(x) + Q(x)$, for all $x \in X$.

Conversely, suppose there exist additive mapping $A : X \to Y$ and a quadratic mapping $Q : X \to Y$ such that $f(x) = A(x) + Q(x)$, for all $x \in X$, then using Theorem 3.1, Theorem 3.2 and (3.25), we arrive at the functional equaiton (1.1). □

4 Generalized Hyers-Ulam Stability of Functional Equation (1.1) in non-Archimedean Fields

Throughout this section, X and Y will be a non-Archimedean field and a complete non-Archimedean field, respectively. Define $D_f : X \times X \to Y$ by

$$\begin{aligned} D_f(x, y) = {} & 4[f(x + 3y) + f(3x + y)] - 9f(x + y) + 15f(x - y) \\ & - 4f(3x) - 10f(x) - 9f(3y) + 35f(y) \end{aligned}$$

for all $x, y \in X$.

Theorem 4.1. Let $\varphi : X \times X \to Y$ be a function such that

$$\lim_{n\to\infty} \left|\frac{1}{9}\right|^n \varphi\left(3^n x, 3^n y\right) = 0 \tag{4.1}$$

for all $x, y \in X$. Suppose that $f : X \to Y$ is an even mapping satisfying the inequality

$$\|D_f(x,y)\| \leq \varphi(x,y) \tag{4.2}$$

for all $x, y \in X$. Then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$\|f(x) - Q(x)\| \leq max\left\{\left|\frac{1}{9}\right|^j \varphi\left(0, 3^j x\right) : j \in \mathbb{N}\cup\{0\}\right\} \tag{4.3}$$

for all $x \in X$.

Proof. Replacing (x, y) by $(0, x)$ in (4.2) and dividing by 9, we get

$$\left\|f(3x) - \frac{1}{9}f(x)\right\| \leq \left|\frac{1}{9}\right|\varphi(0,x) \tag{4.4}$$

for all $x \in X$. Substituting x by $\left(\frac{1}{9}\right)^n x$ in (4.4) and multiplying by $\left|\frac{1}{9}\right|^n$, we have

$$\left\|\frac{1}{9^n}f\left(3^n x\right) - \frac{1}{9^{n+1}}f\left(3^{n+1}x\right)\right\| \leq \left|\frac{1}{9}\right|^n \varphi\left(0, 3^n x\right) \tag{4.5}$$

for all $x \in X$. Thus the sequence $\left\{\frac{1}{9^j}f\left(3^j x\right)\right\}$ is Cauchy by (4.1) and (4.5). Completeness of the non-Archimedean space Y allows us to assume that there exists a mapping Q so that

$$\lim_{n\to\infty} \frac{1}{9^n}f\left(3^n x\right) = Q(x). \tag{4.6}$$

For each $x \in X$ and non-negative integers n, we have

$$\begin{aligned}\left\|\frac{1}{9^n}f\left(3^n x\right) - f(x)\right\| &= \left\|\sum_{j=0}^{n-1}\left\{\frac{1}{9^{j+1}}f\left(3^{j+1}x\right) - \frac{1}{9^j}f\left(3^j x\right)\right\}\right\| \\ &\leq max\left\{\left\|\frac{1}{9^{j+1}}f\left(3^{j+1}x\right) - \frac{1}{9^j}f\left(3^j x\right)\right\| : 0 \leq j < n\right\} \\ &\leq max\left\{\left|\frac{1}{9}\right|^j \varphi\left(0, 3^j x\right) : 0 \leq j \leq n\right\}. \end{aligned} \tag{4.7}$$

Applying (4.6) and letting n to infinity, we find that the inequality (4.3) holds. From (4.1), (4.2) and (4.6), we have for all $x, y \in X$

$$\begin{aligned}\|D_Q(x,y)\| &= \lim_{n\to\infty}\left|\frac{1}{9}\right|^n \left\|D_f\left(3^n x, 3^n y\right)\right\| \\ &\leq \lim_{n\to\infty}\left|\frac{1}{9}\right|^n \varphi\left(3^n x, 3^n y\right) = 0.\end{aligned}$$

Hence the mapping Q satisfies the functional equation (1.1). By Theorem 3.1, the mapping Q is quadratic. Now, let $Q' : X \to Y$ be another quadratic mapping satisfying (4.3). Then we have

$$\begin{aligned}\left\|Q'(x) - Q(x)\right\| &= \lim_{m\to\infty}\left|\frac{1}{9}\right|^m \left\|Q'\left(3^m x\right) - Q\left(3^m x\right)\right\| \\ &\leq \lim_{m\to\infty}\left|\frac{1}{9}\right|^m max\Big\{\left\|Q'\left(3^m x\right) - f\left(3^m x\right)\right\|, \left\|f\left(3^m x\right) - Q\left(3^m x\right)\right\|\Big\} \\ &\leq \lim_{m\to\infty}\lim_{n\to\infty} max\Big\{max\Big\{\left|\frac{1}{9}\right|^{j+m}\varphi\left(0, 3^{j+m}x\right) : m \leq j \leq n+m\Big\}\Big\} \\ &= 0\end{aligned}$$

for all $x \in X$, which proves that Q is unique. Hence the proof is complete. $\square$

Theorem 4.2. *Let $\varphi : X \times X \to Y$ be a function such that*

$$\lim_{n\to\infty} \left|\frac{1}{3}\right|^n \varphi\left(3^n x, 3^n y\right) = 0 \tag{4.8}$$

for all $x, y \in X$. Suppose that $f : X \to Y$ is an odd mapping satisfying the inequality

$$\|D_f(x,y)\| \le \varphi(x,y) \tag{4.9}$$

for all $x, y \in X$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - A(x)\| \le max\left\{\left|\frac{1}{3}\right|^j \varphi\left(0, 3^j x\right) : j \in \mathbb{N} \cup \{0\}\right\} \tag{4.10}$$

for all $x \in X$.

Proof. Replacing (x, y) by $(0, x)$ in (4.9) and dividing by 15, we get

$$\left\|f(3x) - \frac{1}{3}f(x)\right\| \le \varphi(0,x) \tag{4.11}$$

for all $x \in X$. Substituting x by $\left(\frac{1}{3}\right)^n x$ in (4.11) and multiplying by $\left|\frac{1}{3}\right|^n$, we have

$$\left\|\frac{1}{3^n}f\left(3^n x\right) - \frac{1}{3^{n+1}}f\left(3^n x\right)\right\| \le \left|\frac{1}{3}\right|^n \varphi\left(0, 3^n x\right) \tag{4.12}$$

for all $x \in X$. Thus the sequence $\left\{\frac{1}{3^j}f\left(3^j x\right)\right\}$ is Cauchy by (4.8) and (4.12). Completeness of the non-Archimedean space Y allows us to assume that there exists a mapping A so that

$$\lim_{n\to\infty} \frac{1}{3^n}f\left(3^n x\right) = A(x). \tag{4.13}$$

For each $x \in X$ and non-negative integers n, we have

$$\begin{aligned}\left\|\frac{1}{3^n}f\left(3^n x\right) - f(x)\right\| &= \left\|\sum_{j=0}^{n-1}\left\{\frac{1}{3^{j+1}}f\left(3^{j+1}x\right) - \frac{1}{3^j}f\left(3^j x\right)\right\}\right\| \\ &\le max\left\{\left\|\frac{1}{3^{j+1}}f\left(3^{j+1}x\right) - \frac{1}{3^j}f\left(3^j x\right)\right\| : 0 \le j < n\right\} \\ &\le max\left\{\left|\frac{1}{3}\right|^j \varphi\left(0, 3^j x\right) : 0 \le j \le n\right\}.\end{aligned} \tag{4.14}$$

Applying (4.13) and letting n to infinity, we find that the inequality (4.10) holds. From (4.8), (4.9) and (4.14), we have for all $x, y \in X$

$$\begin{aligned}\|D_A(x,y)\| &= \lim_{n\to\infty}\left|\frac{1}{3}\right|^n \left\|D_f\left(3^n x, 3^n y\right)\right\| \\ &\le \lim_{n\to\infty}\left|\frac{1}{3}\right|^n \varphi\left(3^n x, 3^n y\right) = 0.\end{aligned}$$

Hence the mapping A satisfies the functional equation (1.1). By Theorem 3.2, the mapping A is additive. Now, let $A' : X \to Y$ be another additive mapping satisfying (4.10). Then we have

$$\begin{aligned}\|A'(x) - A(x)\| &= \lim_{m\to\infty} \left|\frac{1}{3}\right|^m \|A'(3^m x) - A(3^m x)\| \\ &\le \lim_{m\to\infty} \left|\frac{1}{3}\right|^m max\Big\{ \|A'(3^m x) - f(3^m x)\|, \|f(3^m x) - A(3^m x)\| \Big\} \\ &\le \lim_{m\to\infty} \lim_{n\to\infty} max\Big\{max\Big\{ \left|\frac{1}{3}\right|^{j+m} \varphi\left(0, 3^{j+m}x\right) : m \le j \le n+m\Big\}\Big\} \\ &= 0\end{aligned}$$

for all $x \in X$, which proves that A is unique. Hence the proof is complete. □

Theorem 4.3. *Let $\varphi : X \times X \to Y$ be a function such that*

$$\lim_{n\to\infty} \left|\frac{1}{9}\right|^n \varphi(3^n x, 3^n y) = 0 \qquad and \qquad \lim_{n\to\infty} \left|\frac{1}{3}\right|^n \varphi(3^n x, 3^n y) = 0 \tag{4.15}$$

for all $x, y \in X$. Suppose that $f : X \to Y$ is a mapping satisfying the inequality

$$\|D_f(x,y)\| \le \varphi(x,y) \tag{4.16}$$

for all $x, y \in X$. Then there exist a unique quadratic mapping $Q : X \to Y$ and a unique additive mapping $A : X \to Y$ such that

$$\begin{aligned}&\|f(x) - Q(x) - A(x)\| \\ &\quad \le max\left\{ \left|\frac{1}{9}\right|^j \varphi\left(0, 3^j x\right), \left|\frac{1}{3}\right|^j \varphi\left(0, 3^j x\right) : j \in \mathbb{N} \cup \{0\} \right\}\end{aligned} \tag{4.17}$$

for all $x \in X$.

Proof. Using Theorem 4.1 and Theorem 4.2, we obtain the required results of the theorem. □

Corollary 4.4. *Let $\epsilon \ge 0$ be a constant. Suppose that $f : X \to Y$ is a mapping satisfying the inequality*

$$\|D_f(x,y)\| \le \epsilon \tag{4.18}$$

for all $x, y \in X$. Then there exist a unique quadratic mapping $Q : X \to Y$ and a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - Q(x) - A(x)\| \le \epsilon \tag{4.19}$$

for all $x \in X$.

Proof. The proof is obtained by taking $\varphi(x,y) = \epsilon$, for all $x, y \in X$. Then we have $\varphi\left(0, 3^j x\right) = \epsilon$, for all $x \in X$ and $j \in \mathbb{N} \cup \{0\}$

$$\begin{aligned}\|f(x) - Q(x) - A(x)\| &\le \max\left\{ \left|\frac{1}{9}\right|^j \epsilon, \left|\frac{1}{3}\right|^j \epsilon : j \in \mathbb{N} \cup \{0\} \right\} \\ &\le \epsilon\end{aligned}$$

for all $x \in X$. □

Corollary 4.5. Let $\theta \geq 0$ and $p < 1$ be constants. Suppose that $f : X \to Y$ is a mapping satisfying the inequality

$$\|D_f(x, y)\| \leq \theta \left(\|x\|^p + \|y\|^p\right) \tag{4.20}$$

for all $x, y \in X$. Then there exist a unique quadratic mapping $Q : X \to Y$ and a unique additive mapping $A : X \to Y$ such that

$$\|f(x) - Q(x) - A(x)\| \leq 3^{p-1}\theta \|x\|^p \tag{4.21}$$

for all $x \in X$.

Proof. The required results are obtained by taking $\varphi(x, y) = \theta \left(\|x\|^p + \|y\|^p\right)$, for all $x, y \in X$. Then we have $\varphi\left(0, 3^j x\right) = 3^{jp}\theta \|x\|^p$, for all $x \in X$ and $j \in \mathbb{N} \cup \{0\}$

$$\begin{aligned}\|f(x) - Q(x) - A(x)\| &\leq \max\left\{\left|\frac{1}{9}\right|^j 3^{jp}\theta \|x\|^p, \left|\frac{1}{3}\right|^j 3^{jp}\theta \|x\|^p : j \in \mathbb{N} \cup \{0\}\right\} \\ &\leq \max\left\{3^{(p-2)j}\theta \|x\|^p, 3^{(p-1)j}\theta \|x\|^p : j \in \mathbb{N} \cup \{0\}\right\} \\ &\leq 3^{p-1}\theta \|x\|^p\end{aligned}$$

for all $x \in X$. □

5 Conclusion

Thus, we have obtained the general solution of the mixed type functional equation (1.1) and proved that the generalized Hyers-Ulam stability of the functional equation (1.1) is also stable in non-Archimedean fields.

Competing Interests

The authors declare that no competing interests exist.

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