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Fixed Points of Expansive Type Maps in Cone Metric Space over Banach Algebra

Richa Gupta[1] and Anil Bakhru[2*]

[1]Department of Engineering Mathematics and Research Center, Sarvepalli Radhakrishanan University, Bhopal, 462026, India.
[2]Department of Mathematics, Sardar Vallabhbhai Polytechnic College, College of University of Technology of Madhya Pradesh, Bhopal, 462002, India.

Authors' contributions

This work was carried out in collaboration between both authors. Author AB designed the study, performed the statistical analysis, wrote the protocol and wrote the first draft of the manuscript. Authors AB and RG managed the analyses of the study. Author RG managed the literature searches. Both authors read and approved the final manuscript.

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Abstract

Our aim of this paper is to prove some fixed point and common fixed theorems for contractive type maps in a cone metric space over Banach algebra, which unify, extend and generalize most of the existing relevant fixed point theorems from Jiang et al. [1].

Keywords: Cone metric space; Banach algebra; expansive mapping; fixed point.

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1 Introduction

Fixed point theory has extensive applications in numerous branches of mathematics, digital signal processing, physics, chemistry, Nash equilibrium, economics, optimization, engineering, biology, medical

**Corresponding author: E-mail: anilbakhru@gmail.com;*

sciences, classical analysis, probabilistic analysis, functional analysis, operator theory, theory of non-linear oscillations, general and algebraic topology and statistics with various problems in the theory of differential and integral equations, approximation theory, game theory, and others. To show the existence and uniqueness of fixed points and to determine common fixed points are very popular for researchers in this area. For more details, we refer the reader to [2-10]. The study of expansive mappings is fascinating research area of fixed point theory. The concept of expansive mappings has been introduced in [5]. Fixed point results have been obtained in [11] and [12] for expansive mapping. There are many studies [13,14,15,16,17,18,1] on fixed point theorems in metric spaces for expansive mapping. As a generalization of metric spaces, cone metric spaces were scrutinized by Huang and Zhang in 2007 (see [16]). Afterwards, by omitting the assumption of normality in the theorems of [16], Rezapour and Hamlbarani [3] established some fixed point theorems, as the generalizations and extensions of the analogous results in [16]. Besides, they gave a number of examples to vouch the existence of non-normal cones, which proves that such generalizations are significant. For more details, we refer the reader to [19,20,21,22,23,24,25,26,27,28,29,2,3]. Liu and Xu [30] introduced the notion of cone metric space over Banach algebras, replacing Banach spaces by Banach algebras as the underlying spaces of cone metric space. For more details, we refer the reader to [31,30,32,33,6]. In this paper, we established some fixed point and common fixed theorems for contractive type maps in a cone metric space over Banach algebra, which unify, extend and generalize most of the existing relevant fixed point theorems from Jiang et al. [1] and Daheriya et al. [13].

2 Preliminaries

For the sake of reciter, we shall recollect some fundamental concepts and lemmas. We begin with the following definition as a recall from [30].

Let $\mathcal{A}$ always is a real Banach algebra. Then $\forall\, u, v, w \in \mathcal{A}, \alpha \in \mathbb{R}$, we have

1. $(uv)w = u(vw)$;
2. $u(v + w) = uv + uw$ and $(u + v)w = uw + vw$;
3. $\alpha(uv) = (\alpha u)v = u(\alpha v)$;
4. $\|uv\| \leq \|u\|\|v\|$.

We shall assume that a Banach algebra has a multiplicative identity e such that $eu = ue = u, \forall\, u \in \mathcal{A}$. An element $u \in \mathcal{A}$ is said to be invertible if there is an inverse element $v \in \mathcal{A}$ such that $uv = vu = e$. The inverse of u is denoted by u^{-1}. For more details, we refer the reader to [4].

The following proposition is given in [4].

Proposition 2.1

(see [4]) Let $\mathcal{A}$ be a Banach algebra with a unit e, and $u \in \mathcal{A}$. If the spectral radius $\rho(u)$ of u is less than 1, i.e.,

$$\rho(u) = \lim_{n\to\infty} \|u^n\|^{\frac{1}{n}} = \inf\|u^n\|^{\frac{1}{n}} < 1,$$

then $e - u$ is invertible. Actually,

$$(e - u)^{-1} = \sum_{i=0}^{\infty} u_i.$$

Remark 2.2

From [4] we see that the spectral radius $\rho(u)$ of u satisfies $\rho(u) \leq \|u\|, \forall\, u \in \mathcal{A}$, where $\mathcal{A}$ is a Banach algebra with a unit e.

Remark 2.3

(see [4]) In Proposition 2.1, if the condition "$\rho(u) < 1$" is replaced by "$\|u\| < 1$", then the conclusion remains true.

A subset P of $\mathcal{A}$ is called a cone of $\mathcal{A}$ if

1. P is nonempty closed and $\{\theta, e\} \subset P$;
2. $\delta P + \mu P \subset P$ for all nonnegative real numbers δ, μ;
3. $P^2 = PP \subset P$;
4. $P \cap (-P) = \{\theta\}$,

where θ denotes the null of the Banach algebra $\mathcal{A}$. For a given cone $P \subset \mathcal{A}$, we can define a partial ordering $\preccurlyeq$ with respect to P by $u \preccurlyeq v$ if and only if $v - u \in P$. $u \prec v$ will stand for $u \preccurlyeq v$ and $u \neq v$, while $u \ll v$ will stand for $v - u \in \text{int}P$, where $\text{int}P$ denotes the interior of P. If $\text{int}P \neq \emptyset$, then P is called a solid cone.

The cone P is called normal if there is a number $M > 0$ such that, $\forall\, u, v \in \mathcal{A}$, $\theta \preccurlyeq u \preccurlyeq v \Rightarrow \|u\| \leq M\|v\|$. The least positive number satisfying the above is called the normal constant of P (see [16]).

In the following we always assume that $\mathcal{A}$ is a Banach algebra with a unit e, P is a solid cone in $\mathcal{A}$ and $\preccurlyeq$ is the partial ordering with respect to P.

Definition 2.4

(see [30]) Let X be a nonempty set. Suppose that the mapping $d: X \times X \to \mathcal{A}$ satisfies

1. $\theta \preccurlyeq d(u, v), \forall\, u, v \in X$ and $d(u, v) = \theta \Leftrightarrow u = v$;
2. $d(u, v) = d(v, u), \forall\, u, v \in X$;
3. $d(u, v) \preccurlyeq d(u, w) + d(w, v), \forall\, u, v, w \in X$.

Then d is called a cone metric on X, and (X, d) is called a cone metric space over Banach algebra $\mathcal{A}$.

Definition 2.5

(see [30]) Let (X, d) be a cone metric space over a Banach algebra $\mathcal{A}$, $u \in X$ and let $\{u_n\}_{n=0}^{\infty} \subset X$ be a sequence. Then:

1. $\{u_n\}$ converges to u whenever for each $c \in \mathcal{A}$ with $c \gg \theta$, $\exists$ a natural number N such that $d(u_n, u) \ll c$ for all $n \geq N$. We write $\lim_{n\to\infty} u_n = u$ or $u_n \to u$ $(n \to \infty)$.
2. $\{u_n\}$ is a Cauchy sequence whenever for each $c \in \mathcal{A}$ with $c \gg \theta$, $\exists$ a natural number N such that $d(u_n, u_m) \ll c$ for all $n, m \geq N$.
3. (X, d) is a complete cone metric space over Banach algebra $\mathcal{A}$, if every Cauchy sequence is convergent in X.

Now, we shall appeal to the following lemmas in the sequel.

Lemma 2.6

(see [34]) If E is a real Banach space with a cone P and if $b \preccurlyeq \mu b$ with $b \in P$ and $\theta \leq b < 1$, then $b = \theta$.

Lemma 2.7

(see [2]) If E is a real Banach space with a solid cone P and if $\theta \preccurlyeq x \ll c$ for each $\theta \ll c$, then $x = \theta$.

Lemma 2.8

(see [2])) Let P be a cone in a Banach algebra $\mathcal{A}$ and $k \in P$ be a given vector. Let $\{u_n\}$ be a sequence in P. If for each $c_1 \gg \theta$, there exists N_1 such that $u_n \ll c_1$ for all $n > N_1$, then for each $c_2 \gg \theta$, there exists N_2 such that $ku_n \ll c_2$ for all $n > N_2$.

Lemma 2.9

(see [2]) If E is a real Banach space with a solid cone P and $\{x_n\} \subset P$ is a sequence with $\|x_n\| \to 0$ ($n \to \infty$) then for any $\theta \ll c$, $\exists\, N \in \mathbb{N}$ such that, for any $n > N$, we have $x_n \ll c$, i.e. x_n is a c-sequence

Finally, let us recall the concept of generalized Lipschitz mapping defining on the cone metric space over Banach algebras, which is introduced in [30].

Remark 2.10

(see [4]) If $\rho(u) < 1$, then $\|u\|^n \to 0$ ($n \to \infty$).

Lemma 2.11

(see [4]) Let $\mathcal{A}$ be a Banach algebra with a unit $e, h, k \in \mathcal{A}$. If h commutes with k, then

$$\rho(h + k) \leq \rho(h) + \rho(k),$$

$$\rho(hk) \leq \rho(h)\rho(k).$$

Lemma 2.12

(see [4]) If E is a real Banach space with a solid cone P

(1). If $a_1, a_2, a_3 \in E$ and $a_1 \preccurlyeq a_2 \ll a_3$, then $a_1 \ll a_3$.
(2). If $a_1 \in P$ and $a_1 \ll a_3$ for each $a_3 \gg \theta$, then $a_1 = \theta$.

Lemma 2.13

(see [6]) Let P be a solid cone in a Banach algebra $\mathcal{A}$. Suppose that $h \in P$ and $\{x_n\} \subset P$ is a c-sequence. Then $\{hx_n\}$ is a c-sequence.

Proposition 2.14

(see [6]) Let P be a solid cone in a Banach space $\mathcal{A}$ and let $\{u_n\}, \{v_n\} \subset X$ be sequences. If $\{u_n\}$ and $\{v_n\}$ are c-sequences and $\gamma, \delta > 0$ then $\{\gamma u_n + \delta u_n\}$ is a c-sequence.

Proposition 2.15

(see [6]) Let P be a solid cone in a Banach algebra $\mathcal{A}$ and let $\{u_n\} \subset P$ is a sequence. Then the following conditions are equivalent:

1. $\{u_n\}$ is a c-sequence.
2. For each $c \gg \theta$ there exists $n_0 \in \mathbb{N}$ such that $u_n \prec c$ for $n \geq n_0$.
3. For each $c \gg \theta$ there exists $n_1 \in \mathbb{N}$ such that $u_n \preccurlyeq c$ for $n \geq n_1$.

Lemma 2.16

(see [4]) Let $\mathcal{A}$ be a Banach algebra with a unit $e, h \in \mathcal{A}$, then $\lim_{n\to\infty}\|h^n\|^{\frac{1}{n}}$ exists and the spectral radius $\rho(h)$ satisfies

$$\rho(h) = \lim_{n\to\infty}\|h^n\|^{\frac{1}{n}} = \inf\|h^n\|^{\frac{1}{n}}.$$

If $\rho(h) < |\lambda|$, then $(\lambda e - h)$ is invertible in $\mathcal{A}$; moreover,

$$(\lambda e - h)^{-1} = \sum_{i=0}^{\infty} \frac{h^i}{\lambda^{i+1}}.$$

where λ is a constant.

Lemma 2.17

(see [4]) Let $\mathcal{A}$ be a Banach algebra with a unit e and $h \in \mathcal{A}$. If λ is a complex constant and $\rho(h) < |\lambda|$, then

$$\rho\left((\lambda e - h)^{-1}\right) \leq \frac{1}{|\lambda| - \rho(h)}.$$

Lemma 2.18

(see [4]) Let $\mathcal{A}$ be a Banach algebra with a unit e and P be a solid cone in $\mathcal{A}$. Let $h \in \mathcal{A}$ and $u_n = h^n$. If $\rho(h) < 1$, then $\{u_n\}$ is a c-sequence.

Definition 2.19

(see [1]) Let (X, d) be a complete cone metric space over Banach algebra $\mathcal{A}$ and P be a cone in $\mathcal{A}$. Then $f: X \to X$ is said to be generalized expansive mapping, if

$$d(fu, fv) \succcurlyeq h\, d(u, v).$$

for all $u, v \in X$, where $h, h^{-1} \in P$ are generalized constants with $\rho(h^{-1}) < 1$.

3 Main Results

Theorem 3.1

Let (X, d) be a cone metric space over Banach algebra $\mathcal{A}$ and P be an underlying solid cone in $\mathcal{A}$, where $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, k, -\mathfrak{a} \in P$ be generalized Lipschitz constants with

$$\rho\left((e + k - \mathfrak{a})(\mathfrak{b} + \mathfrak{c} - k)^{-1}\right) < 1.$$

Let f be a surjective self-map of X satisfies the expansive condition

$$d(fu, fv) + k\,[d(u, fv) + d(v, fu)] \succcurlyeq \mathfrak{a}\, d(u, fu) + \mathfrak{b}\, d(v, fv) + \mathfrak{c}\, d(u, v), \tag{3.1}$$

for all $u, v \in X$, and then f has a unique fixed point in X.

Proof Since f is surjective, for each $u_0 \in X$, there exists $u_1 \in X$ such that $fu_1 = u_0$. Continuing this process, we can define $\{u_n\}$ by

$$u_n = fu_{n+1} \; (n = 1,2,3, \ldots).$$

If $u_n = u_{n+1}$ for some n, then we say that u_n is a fixed point of f. Therefore, we suppose that no two consecutive terms of the sequence $\{u_n\}$ are equal. Thus, by (3.1), for any natural number n, on the one hand, we obtain

$$d(fu_{n+1}, fu_{n+2}) + k[d(u_{n+1}, fu_{n+2}) + d(u_{n+2}, fu_{n+1})]$$
$$\succcurlyeq \mathfrak{a}d(u_{n+1}, fu_{n+1}) + \mathfrak{b}d(u_{n+2}, fu_{n+2}) + \mathfrak{c}d(u_{n+1}, u_{n+2})$$

That is,

$$d(u_n, u_{n+1}) + k[d(u_{n+1}, u_{n+1}) + d(u_{n+2}, u_n)] \succcurlyeq \mathfrak{a}d(u_{n+1}, u_n) + \mathfrak{b}d(u_{n+2}, u_{n+1}) + \mathfrak{c}d(u_{n+1}, u_{n+2}).$$

Hence,

$$\mathfrak{a}d(u_{n+1}, u_n) + \mathfrak{b}d(u_{n+2}, u_{n+1}) + \mathfrak{c}d(u_{n+1}, u_{n+2}) \preccurlyeq d(u_n, u_{n+1}) + k[d(u_{n+2}, u_{n+1}) + d(u_{n+1}, u_n)].$$

The last inequality gives

$$(\mathfrak{b} + \mathfrak{c} - k)d(u_{n+2}, u_{n+1}) \preccurlyeq (e + k - a)d(u_{n+1}, u_n). \tag{3.2}$$

Put $(\mathfrak{b} + \mathfrak{c} - k) = r$, then

$$rd(u_{n+2}, u_{n+1}) \preccurlyeq (e + k - a)d(u_{n+1}, u_n). \tag{3.3}$$

Since r is invertible, to multiply r^{-1} on both sides of (3.3),

$$d(u_{n+2}, u_{n+1}) \preccurlyeq hd(u_{n+1}, u_n). \tag{3.4}$$

where $h = (e + k - a)(\mathfrak{b} + \mathfrak{c} - k)^{-1}$. We get

$$d(u_{n+2}, u_{n+1}) \preccurlyeq hd(u_{n+1}, u_n) \preccurlyeq h^2 d(u_n, u_{n-1}) \preccurlyeq \cdots \preccurlyeq h^{n+1} d(u_1, u_0). \tag{3.5}$$

So by the triangle inequality and $\rho(h) < 1$, for all $m > n$, we see

$$\begin{aligned} d(u_m, u_n) &\preccurlyeq d(u_m, u_{m-1}) + d(u_{m-1}, u_{m-2}) + d(u_{n+1}, u_n) \\ &\preccurlyeq (h^{m-1} + h^{m-2} + \cdots + h^{n+1} + h^n)d(u_1, u_0) \\ &= (e + h + h^2 + h^3 + \cdots + h^{m-n-1})h^n d(u_1, u_0) \\ &\preccurlyeq (\textstyle\sum_{i=0}^{\infty} h^i)h^n d(u_1, u_0) \\ &\preccurlyeq h^n (e - h)^{-1} d(u_1, u_0). \end{aligned} \tag{3.6}$$

By Lemma 2.9 and the fact that $\|h^n(e - h)^{-1} d_c(u_1, u_0)\| \to 0 (n \to \infty)$ because of Remark 2.10, $\|h^n\| \to 0 \; (n \to \infty)$, it follows that for any $c \in \mathcal{A}$ with $\theta \ll c$, there exists N such that for all $m > n > N$, we have

$$d(u_m, u_n) \preccurlyeq h^n (e - h)^{-1} d(u_1, u_0) \ll c. \tag{3.7}$$

which implies that $\{u_n\}$ is a Cauchy sequence. By the complete-ness of X, there exists $u^* \in X$ such that $u_n \to u^*(n \to \infty)$. Consequently, we can find an $u^{**} \in X$ such that $fu^{**} = u^*$. Now we show that $u^* = u^{**}$. In fact,

$$d(u^*, u_n) = d(fu^{**}, fu_{n+1})$$

$$\succcurlyeq \mathfrak{a}d(u^{**}, fu^{**}) + \mathfrak{b}d(u_{n+1}, fu_{n+1}) + \mathfrak{c}d(u^{**}, u_{n+1}) - k[d(u^{**}, fu_{n+1}) + d(u_{n+1}, fu^{**})]$$

$$\succcurlyeq \mathfrak{a}d(u^{**}, u^*) + \mathfrak{b}d(u_{n+1}, u_n) + \mathfrak{c}d(u^{**}, u_{n+1}) - k\,[d(u^{**}, u_n) + d(u_{n+1}, u^*)]. \tag{3.8}$$

Since the tringle inequality, we have

$$d(u^{**}, u_{n+1}) - d(u_{n+1}, u^*) \preccurlyeq d(u^{**}, u^*),$$

$$d(u^*, u_n) \preccurlyeq d(u^*, u_{n+1}) + d(u_{n+1}, u_n),$$

and

$$d(u^{**}, u_n) \preccurlyeq d(u^{**}, u_{n+1}) + d(u_{n+1}, u_n).$$

it follows that

$$-kd(u_{n+1}, u^*) + \mathfrak{b}d(u_{n+1}, u_n) + \mathfrak{c}d(u^{**}, u_{n+1})$$

$$\preccurlyeq d(u^*, u_{n+1}) + d(u_{n+1}, u_n) - \mathfrak{a}d(u^{**}, u_{n+1}) + \mathfrak{a}d(u_{n+1}, u^*) + kd(u^{**}, u_{n+1}) + kd(u_{n+1}, u_n).$$

which implies that

$$(\mathfrak{a} + \mathfrak{c} - k)d(u^{**}, u_{n+1}) \preccurlyeq (e + \mathfrak{a} + k)d(u_{n+1}, u^*) + (e - b + k)d(u_{n+1}, u_n). \tag{3.9}$$

Since $\mathfrak{a} + \mathfrak{c} - k = r$ is invertible, we have

$$d(u^{**}, u_{n+1}) \preccurlyeq r^{-1}\big((e + \mathfrak{a} + k)d(u_{n+1}, u^*) + (e - b + k)d(u_{n+1}, u_n)\big). \tag{3.10}$$

Owing to $u_n \to u^*(n \to \infty)$, it follows by Lemma 2.8 that for any $c \in \mathcal{A}$ with $\theta \ll c$, there exists N such that for any $n > N$,

$$r^{-1}\big((e + \mathfrak{a} + k)d(u_{n+1}, u^*) + (e - b + k)d(u_{n+1}, u_n)\big) \ll c. \tag{3.11}$$

hence $d(u^{**}, u_{n+1}) \ll c$. Since the limit of a convergent sequence in cone metric space over Banach algebra is unique, we have $u^* = u^{**}$, i.e., u^* is a fixed point of f.

Finally, we prove the uniqueness of the fixed point. In fact, if v^* is another common fixed point of f, that is, $fv^* = v^*$, from (3.1), we have

$$d(fu^*, fv^*) + k\,[d(u^*, fv^*) + d(v^*, fu^*)] \succcurlyeq \mathfrak{a}d(u^*, fu^*) + \mathfrak{b}d(v^*, fv^*) + \mathfrak{c}d(u^*, v^*)$$

$$\implies \quad d(u^*, v^*) + k[d(u^*, v^*) + d(v^*, u^*)] \succcurlyeq \mathfrak{a}d(u^*, u^*) + \mathfrak{b}d(v^*, v^*) + \mathfrak{c}d(u^*, v^*).$$

The last inequality gives

$$(c - 2k - e)d(u^*, v^*) \preccurlyeq \theta. \tag{3.12}$$

which means $d(u^*, v^*) = \theta$, which implies that $u^* = v^*$, a contradiction. Hence the fixed point is unique.

Corollary 3.1

Let (X, d) be a cone metric space over Banach algebra $\mathcal{A}$ and P be an underlying solid cone in $\mathcal{A}$, where $\mathfrak{c} \in P$ be generalized Lipschitz constants with $\rho(\mathfrak{c}^{-1}) < 1$. Let f be a surjective self-map of X satisfying the expansive condition

$$d(fu, fv) \succcurlyeq \mathfrak{c}d(u, v). \tag{3.13}$$

for all $u, v \in X$, and then f has a unique fixed point in X.

Proof If we put $k, \mathfrak{a}, \mathfrak{b} = \theta$ in Theorem 3.1, then we get the above Corollary 3.1.

Corollary 3.2

Let (X, d) be a cone metric space over Banach algebra $\mathcal{A}$ and P be an underlying solid cone in $\mathcal{A}$, where $\mathfrak{c} \in P$ be generalized Lipschitz constants with $\rho(\mathfrak{c}^{-1}) < 1$. Let f be a surjective self-map of X, and suppose that there exists a positive integer n satisfying

$$d(f^n u, f^n v) \succcurlyeq \mathfrak{c}d(u, v). \tag{3.14}$$

for all $u, v \in X$, and then f has a unique fixed point in X.

Proof From Corollary 3.1, f^n has a unique fixed point z. But $f^n(fu) = f(f^n u) = fu$, so fu is also a fixed point of f^n. Hence $fu = u$, u is a fixed point of f . Since the fixed point of f is also a fixed point of f^n, the fixed point of f is unique.

Corollary 3.3

Let (X, d) be a cone metric space over Banach algebra $\mathcal{A}$ and P be an underlying solid cone in $\mathcal{A}$, where $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, -\mathfrak{a} \in P$ be generalized Lipschitz constants with $\rho\big((e - \mathfrak{a})(\mathfrak{b} + \mathfrak{c})^{-1}\big) < 1$. Let f be a surjective self-map of X satisfies the expansive condition

$$d(fu, fv) \succcurlyeq \mathfrak{a}\, d(u, fu) + \mathfrak{b}\, d(v, fv) + \mathfrak{c}\, d(u, v). \tag{3.15}$$

for all $u, v \in X$, and then f has a unique fixed point in X.

Proof If we put $k = \theta$ in Theorem 3.1, then we get the above Corollary 3.3.

4 Example

Example 4.1

Let $\mathcal{A} = \left\{p = (p_{ij})_{3\times 3} \mid p_{ij} \in \mathbb{R}, 1 \le i, j \le 3\right\}$ and $\|p\| = \frac{1}{3}\sum_{1\le i,j\le 3}|p_{ij}|$. Then the set

$$P = \{p \in \mathcal{A} | p_{ij} \ge 0 \text{ and } 1 \le i, j \le 3\}$$

is a normal cone in $\mathcal{A}$. Let $X = \{1,2,3\}$. Define $d : X \times X \to \mathcal{A}$ by

$$d(1,1) = d(2,2) = d(3,3) = \theta$$

$$d(1,2) = d(2,1) = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 4 & 5 & 6 \end{pmatrix}$$

$$d(1,3) = d(3,1) = \begin{pmatrix} 2 & 4 & 6 \\ 0 & 0 & 0 \\ 3 & 4 & 5 \end{pmatrix}$$

$$d(2,3) = d(3,2) = \begin{pmatrix} 2 & 4 & 6 \\ 0 & 0 & 0 \\ 3 & 4 & 5 \end{pmatrix}$$

We find that (X, d) is a solid cone metric space over Banach algebra $\mathcal{A}$. Let $f: X \to X$ be a mapping defined by

$$f1 = 2, f2 = 1, f3 = 3,$$

and let $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, k \in P$ be defined by

$$\mathfrak{a} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{10} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \mathfrak{b} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{10} & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\mathfrak{c} = \begin{pmatrix} \frac{4}{5} & 0 & 0 \\ 0 & \frac{4}{5} & 0 \\ 0 & 0 & \frac{4}{5} \end{pmatrix}, \qquad k = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Then

$$d(fu, fv) + k\,[d(u, fv) + d(v, fu)] \succcurlyeq \mathfrak{a}\, d(u, fu) + \mathfrak{b}\, d(v, fv) + \mathfrak{c}\, d(u, v)$$

where $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, k, -\mathfrak{a} \in P$ are generalized constants. It is easy to prove that

$$\|(e + k - \mathfrak{a})(\mathfrak{b} + \mathfrak{c} - k)^{-1}\| < 1$$

which imply

$$\rho\big((e + k - \mathfrak{a})(\mathfrak{b} + \mathfrak{c} - k)^{-1}\big) < 1.$$

Clearly, all conditions of Theorem 3.1 are fulfilled. Hence f has a unique fixed point $u = 3 \in X$.

5 Conclusion

In this paper, we proved some fixed point theorems under expansive type conditions in cone metric space over Banach algebras. Our results are more general than that of the results of Jiang et al. [2]. This result can be extended to other spaces. Example is constructed to support our result.

Competing Interests

Authors have declared that no competing interests exist.

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