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A Clear Conception of Zero and Infinity with Practical Illustrations

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The sole author designed, analyzed, interpreted and prepared the manuscript.

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Abstract

This paper presents a clear conception of zero and infinity and furnishes some instances to show how this may be employed in Physics.

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1 Introduction

Few mathematical subjects have hatched more impetuous logomachy among mathematicians and scientists than the theme of zero and infinity. Temples of science have sundered and verbal wars have been even fought over the issues that encompass the concepts of zero and infinity.

Many modern mathematicians are not contented with today's interpretation of zero and infinity. In fact, such erudite mathematicians as Barukcic [1], Saitoh[2], Bergstra[3], Anderson [4], Czajko

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[5], [6], [7], and Mwangi [8] have written volumes in order to clarify the notions of zero and infinity. Therefore, the goal of this paper is to provide a lucid conception of these two giant notions which have plagued mathematical analysis since time immemorial. But the paper is not written for every particular category of readers; it is concocted for those who wish to use the notions of zero and infinity as an instrument of investigation or in the attainment of further knowledge.

Though I have made it a point to be laconic, I have constantly endeavoured to make all things as overt as the subject matter will permit. I hope that the reader will need only a moderate effort to apprehend the object and the principles laid down in this paper.

As would be the case with any facet of mathematics, the notions of zero and infinity cannot be expounded fully in a brief paper. This paper thus furnishes a prominent complement to my other papers in this journal. In particular, since the arithmetic of zero and infinity will not be fully discussed here, reading this paper together with the papers [9] and [10] is recommended.

The residuum of this paper is parted into six sections. Section 2 discusses the history of infinity and its relation to zero. Sections 3 and 4 deal with those ideas which have to do with zero and infinity respectively. Section 5 is concerned with the violation of mathematical provisos relating to division by zero using the novel ideas on zero and infinity discussed in Section 3 and 4. Therefore, the essence of this Section is to convince the reader of the soundness of these ideas. The purpose of Sections 6 and 7 is to provide practical illustrations concerned with zero and infinity.

2 History of Infinity in Relation to Zero

Before we enter into the heart of the subject of zero and infinity, we present the historical profile of their relationship.

The histories of zero and infinity [11], [12] are so extensive that a great volume may be written concerning them. These notable concepts are hardly ever construed by the finite mind and, as we know them, involve immeasurable dimensions and are of great utility in the analysis of curvilinear objects.

2.1 Greek Mathematicians

The great story of infinity begins with the ancients who had immense difficulties with its meaning and use, and could not, therefore, admit it into their geometry.

2.1.1 Anaximander

The pre-Socratic Greek philosopher Anaximander was perhaps the first to pen down the idea of infinity. He employed the term *apeiron* to refer to the infinite or boundless.

2.1.2 Zeno

Infinity was a tangled mess because of the entrance of paradoxes and strange as it may appear at first, the Greeks' way of imagining infinity tended to further complication, for Zeno of Elea (c.490 BC-c.430 BC), a pre-Socratic Greek philosopher of southern Italy, came up with some renowned paradoxes that the rectilinear (the straight) and the curvilinear (the curved) are incompatible. For instance, he claimed that an arrow flying in the air can never strike its target since it must of necessity first transverse half the way to the target and then half of the rest distance and so on.

2.1.3 Aristotle

The students of the school of Pythagoras' attempted to resolve this paradox but their attempts were a total fiasco, and Aristotle, in his *Physics*, fought the notion of actual infinity (completed infinity) but championed the concept of potential infinity (uncompleted infinity). In keeping with him, the arrow strikes the target on the ground that space is not divided into portions actually infinite in number. His philosophy was that infinity is not a number but a concept connoting endlessness. For instance, the natural numbers $1, 2, 3, \ldots$ are potentially infinite; they go on without an end. The list of natural numbers has no end or boundary; we can never, given any finite time, reach the end of the sequence of natural numbers. In his illustrious Book III of *Physics* he commented:-

> Our account does not rob the mathematicians of their science, by disproving the actual existence of the infinite in the direction of increase, in the sense of the untransversable. In point of fact they do not need the infinite and do not use it. They postulate only that the finite straight line may be produced as far as they wish.

Thus, Aristotle banned the use of infinite numbers in the science of calculations with numbers.

2.2 Indian Mathematicians

In ancient Indian writings, infinity is considered as a number equal to the quotient of the division of a finite number by zero or nothing.

2.2.1 Brahmagupta

Brahmagupta was the first to allow division by zero. First, he treated zero or nothing as a number. In his book *Brahmasphuta sidhanta*, Brahmagupta gave the formal definition of zero:

> . . . positive (*dhana*) and negative (*rna*), if they are equal, is zero (*kham*).

He then defined the operations of addition, subtraction, and multiplication involving zero:

> The sum of a negative and zero is a negative, of a positive and zero is positive and of two zeros, zero (*sunya*). . . . Negative subtracted from zero is positive, and positive from zero negative. Zero subtracted from negative is negative, from positive is positive, and from zero is zero . . . . The product of zero and a negative, of zero and a positive, of two zeros is zero.

By treating zero 0 as a number, he considered the division of any finite number by zero as a number:

> Positive or negative, divided by zero, is a fraction with zero for denominator. Zero divided by negative or positive either zero or is expressed by a fraction with zero as numerator and the finite quantity as denominator. . . . zero divided by zero is zero . . . .

2.2.2 Mahavira

Brahmagupta's compatriots who came after him took up his enigmatic concept of zero denominator and attempted to clarify it further. Thus Mahavira wrote

> A number remains unchanged when it is divided by zero.

By this Mahavira was suggesting that the finite number acting as numerator of *taccheda* is not partitioned as the dividing number is absolute nothing. Thus *taccheda* has no determinate quotient.

2.2.3 Sripati

Sripati (1039–1056) substituted the more elucidating term *khahara* for *taccheda*, for *khahara* signifies that which has zero (*kha*) as the divisor.

> A number when multiplied by zero becomes zero, and when divided by zero becomes *khahara*.

2.2.4 Bhaskara II

Five hundred years later Bhaskara II modified and advanced these ideas of Brahmagupta and asserted in his *Lilavati* that

> When a number is added to zero the result is that number. The square, & c. of zero is zero. A number divided by zero has zero for its divisor. When a number is multiplied by zero the product is zero; but in case any operation remains to be done, zero is merely conceived to be the multiplier, and when zero also becomes the divisor, the number is considered unchanged.

Concerning the product of zero, Bhaskara gave the following principle: if the product of the finite number a and the quantitative zero 0, i.e $a \cdot 0$, appears as the result of an operation, we should take $a \cdot 0$ equal to the qualitative zero 0, that is $a \cdot 0 = 0$, but if there remains an operation involving the quantitative zero 0 as a divisor to be performed, we should calculate with the original form of expression $a \cdot 0$ but with the understanding that

$$\frac{a \cdot 0}{0} = a.$$

He demonstrated its significance with an instance and concluded that his approach is of great utility in the study of heavenly bodies.

In his *Bijaganita* he commented on the nature of $a/0$

> . . . a quantity, divided by zero, becomes a fraction the denominator of which is zero.

He likened the nature of this fraction to the nature of the infinite God, thereby suggesting that the fraction is a number representing a magnitude endless in extent:

> A quantity divided by zero becomes a fraction the denominator of which is zero. This fraction is termed an infinite quantity (*khahara*). In this quantity consisting of that which has zero for its divisor, there is no alteration, though many may be inserted or extracted; as no change takes place in the infinite and immutable God when worlds are created or destroyed, though numerous orders of beings are absorbed or put forth.

3 On Zeros

3.1 Two Models of Subtraction

In their work [13], the mathematics educationists, Usiskin and Bell, differentiate two models of the operation of subtraction, the *take-away model* and the *comparison model*, leading to two unlike concepts for the result: the *remainder* and the *difference*.

Definition 3.1. The take-away model is a model which claims that a remainder results when a numerical quantity is taken away from another numerical quantity.

Definition 3.2. The comparison model is one which claims that a difference results when two numerical quantities are compared.

Take these two subtraction problems in arithmetic:

I have 7 apples and eat 4 of them. How many apples remain?

and

I have 7 apples and Luke has 4 apples. What is the difference in the number of apples?

Both kind of problems can be expressed with the same number sentence, namely

$$7 - 4 = 3.$$

However, the first kind requires a 'take away' interpretation whereas the second kind needs a 'comparism' interpretation. In the take-away subtraction, the number 3 is the remainder when 4 is taken away from 7 whereas in the comparison subtraction, the number 3 is the difference of 7 and 4.

3.2 Two Models of Absolute Nothing

From the two models of subtraction, two models of *nothing* are inferred-the **takeaway absolute nothing** and the **comparative absolute nothing**.

3.2.1 Takeaway Absolute Nothing

Definition 3.3. The takeaway absolute nothing or simply the takeaway nothing is the void remainder which arises when a numerical quantity is taken away from itself.

The takeaway nothing implies that the quantity taken away from itself has been completely removed, leaving behind emptiness, void or a blank. In mathematics, we use the symbol 0, called **zero**, to signify this nothing. Properly speaking, the takeaway nothing requires no symbol for its denotation because it is the *absence of the quantity taken away.*

If there are c books in a bag and we take away the c books, we are left with no book in the bag and we write 0 to represent "no (absence of) books". The equation $c - c = 0$ means the c books are removed from the bag and as a result there is 0 (absolutely no) book left.

The takeaway nothing occurs during the simplification of expressions or equations. Let us propose the simplification of the expression

$$(2x - 3)^2 + 12x.$$

The simplification goes thus:

$$(2x - 3)^2 + 12x = 4x^2 - 12x + 12x + 9 = 4x^2 + 0 + 9 = 4x^2 + 9.$$

The symbol 0 is omitted forever and can never be reused in further operation because it merely represents absence of $12x$, for $-12x$ and $+12x$ eliminate each other completely.

3.2.2 Comparative Absolute Nothing

Definition 3.4. Comparative absolute nothing or simply comparative nothing is the numerical zero arising from the difference of two quantities when one quantity is made exactly equal to the other.

The quantity which is made equal to the other is called a *variable* and the other quantity to which the variable is made equal is termed a *constant.* When a variable is assigned a value equal to a constant, the difference of the variable and the constant has been made comparative nothing. Whenever a comparative nothing emerges, neither the variable nor the constant is removed or taken away as in the case of takeaway nothing. Thus comparative nothing implies that the minuend and the subtrahend do not cancel out each other. Since the minuend and subtrahend are not eliminated even though their difference is nothing, both quantities, and hence the comparative nothing, are susceptible of further operations. The takeaway nothing is incapable of further operations for the sole reason that both the minuend and subtrahend of the takeaway subtraction which gives birth to it cancel out each other and are thus eliminated.

If there are x (variable number of) books in bag A and c (fixed number of) books in bag B and the number of books in A is made equal to c, then the difference in the number of books in the two bags A and B is comparatively absolute nothing, and we say there is "no (absence of) difference in the number of books" when x is made equal to c and write this as $c - c = 0$. Here, the expression $c - c$ does not stand for the removal of c books from any of the bags but that the number c of books in bag A differs from the number c of books in bag B by absolute nothing.

From the instance adduced, we infer that comparative nothing emerges during the evaluation of expressions wherein a value is substituted for a variable. For example, let

$$y = 4(x - 2)$$

and

$$z = \frac{y}{x - 2}.$$

We wish to find the values of y and z when $x = 2$. Fulfilling this condition, we have

$$y = 4(2 - 2) = 4 \cdot 0.$$

Thus y is the zero $4 \cdot 0$. We employ this zero in the next operation as it is comparative nothing. We use $4 \cdot 0$ instead of 0 in further operation because both the minuend (2) and subtrahend (2) in the comparative subtraction $2 - 2$ are not eliminated. The non-expandable expression $4(2 - 2)$ implies that $2 - 2$ (the *non-void*) is to be taken four times. But, instead of using $4(2 - 2)$ in the next operation we decide to use the symbol 0 in place of $2 - 2$ to inform us that the difference $2 - 2$ is nothing.

Let us now employ $y = 4 \cdot 0$ in the next operation. Hence, we have

$$z = \frac{4 \cdot 0}{0} = 4.$$

I guess the reader notices that the division of the zero $4 \cdot 0$ by the zero 0 is equal to the finite number 4. Perhaps he has been instructed in the way of orthodox mathematics, that the division of nothing by nothing is indeterminate. I must inform the reader here that the division just carried out is not of takeaway nothing by takeaway nothing, for this is impossible; we cannot conceive how absence of quantity can be divided by absence of quantity to give rise to something. But by 0 here we mean the comparative nothing which is not the *void* or mere blank. The zero 0, as we have already noted, is the symbol representing the comparative subtraction $2 - 2$ which is incapable of vanishing since the minuend, 2, and the subtrahend, 2, ever remain, for they do not cancel out or eliminate each other to leave an empty space or blank. Thus, $2 - 2$ remains $2 - 2$ for ever. Hence

$$\frac{2 - 2}{2 - 2} = 1$$

or

$$\frac{0}{0} = 1.$$

There is another demonstration of the validity of this identity and this will be the discussion of a later section.

Let us examine further what is really our meaning when we write $c - c = 0$ for the evaluation of $x - c$ at $x = c$. We do not mean that $c - c$ is the elimination of c; for the minuend c is the value of the variable x which coincides with the constant c which is the subtrahend. To illustrate this point, we consider Fig. 1. If the distance between the fixed point P and the variable point Q is Δx, then $\Delta x = x - c$.

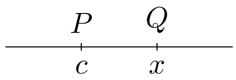

Fig. 1.

Suppose point Q is moved towards point P which is fixed. When Q coincides with P, the variable x coincides with the fixed number c, that is $x = c$. Thus we have

$$\Delta x = c - c = 0.$$

Here $c - c$ is not the elimination of the numbers c. In fact, the two points P and Q still exist and are coincidental points. Thus the expression $c - c$ means that there is no (absence of) difference in value of the variable x and the fixed number or constant c when the variable equals the constant.

Since the two models of absolute nothing discussed above differ considerably, we must avoid confounding the symbol 0 used for takeaway nothing with its use for comparative nothing. To achieve this, we shall, in this work, use the symbol to represent only the takeaway nothing. Thus

$$x - x = 0, \quad c - c = 0, \quad f(x) - f(x) = 0$$

where c is a number, x is a variable or an unknown, and $f(x)$ is any function of x, e.g. $\sin x$, $\ln x$, etc. We shall, however, use the bold–faced symbol $\mathbf{0}$ to represent a unit of comparative nothing.

Little space remains for us to consider the link between the zero $\mathbf{0}$ and the zero 0. Only a few details can therefore be mentioned.

Definition 3.5. Comparative nothing is equivalent to the takeaway nothing 0.

If we start with the equation $x = c$, and subtract c from both sides, we shall have $x - c = c - c$ which becomes $x - c = 0$ where 0, as we have instituted, is the takeaway nothing. For the equation $x - c = 0$ to hold good, x must equal c. Putting c in place of x in the equation furnishes $c - c = 0$ which becomes $\mathbf{0} = 0$, that is, the unit compative nothing is equivalent to the takeaway nothing.

Again, we consider the equation $x^2 = 1$. If we remove 1 from the right-hand side by adding -1 to both sides, we have

$$x^2 - 1 = 0$$

which becomes

$$(x + 1)(x - 1) = 0.$$

If we set $x = 1$, we obtain

$$(\mathbf{1} + 1)(\mathbf{1} - 1) = 0$$

which becomes $2 \cdot \mathbf{0} = 0$. Similar arguments show that for any finite number a and positive number n,

$$a \cdot \mathbf{0}^n = 0.$$

3.3 Are Negative Numbers Less Than Absolute Nothing?

Mathematicians say that numbers with a positive sign represent quantities greater than *absolute nothing*, while those with a negative sign, such as −3, represent quantities less than *absolute nothing.* The phrase, less than nothing, however, cannot convey an intelligible idea with any signification that would be attached to it in the ordinary use of language. *Nothing* implies **no magnitude** [14]. The idea of negative numbers which are clearly quantities, however, would be properly expressed by saying that negative quantities are numbers with effects opposite to those of positive quantities. If we take any number, for instance 10, and add to it its additive inverse, say the number −10, we get *nothing*; the positive effect neutralizes the negative effect to give rise to *no magnitude.* Neutralization or cancellation occurs because both quantities have equal magnitude in opposite orientations.

In integer number line, the mark indicated by 0 represents the transition mark through which positive numbers pass over to negative numbers and *vice versa.* If one walks alone the number line towards 0, one moves from positive quantities with large magnitudes to positive quantities with small magnitudes. When one reaches the mark 0, we say that one has reached the point of *no magnitude* which is *nothing.* If one continues with one's movement alone the integer number line, one passes from negative quantities with small magnitudes to negative quantities with large magnitudes.

Many concrete quantities are capable of existing in these two opposite states. Thus, in financial transactions, we may have gains or losses; in the thermometer, we may have temperatures placed above or below *nothing*; etc. The signs + and −, besides indicating the operations of addition and subtraction, are also used in mathematics to distinguish between the opposite states of magnitudes like the above. Thus, in financial transactions, we may indicate gains or assets by the sign + and losses or liabilities by the sign −. For example, the statement that a man's property is −\$500 means that he has debts or liabilities to the amount of 500. Again, in the thermometer, we may indicate temperatures placed above *nothing* by the sign +, and temperatures placed below *nothing* by the sign −. For example, +50 degree means 50 above *nothing*, and −20 degree means 20 below *nothing.* It is immaterial which state or condition is indicated by the positive sign, but having at the commencement of an investigation indicated a certain condition by the positive sign and the opposite condition by the negative sign, we must retain the same notation throughout. If the positive and negative states of any concrete magnitude are expressed without reference to the unit, the results are called positive and negative numbers, respectively. Thus, in +5 and −3, +5 is a positive number, and −3 a negative number. If no sign is expressed, the number is understood to be positive. The signs + and −, when used to indicate the positive and negative states of numbers respectively, are called *signs of opposition.* The attachment of the sign − to the negative numbers is just a way to convey how much something is opposite to the direction of positive numbers. Traders use negative numbers to convey loss whereas they employ positive numbers to convey gain. They use *zero* as a number representing *nothing*, that is neither gain nor loss. *Zero*, whether positive or negative, is a number representing neutrality or unbiasedness towards positive or negative numbers.

We throw more light on the subject we are now discussing. Positive numbers represent magnitudes of something e.g. length whereas negative numbers represent magnitudes of the same thing but in the opposite direction to the positive numbers. Negative numbers represent how much of something is opposite and antithetical to whatever positive numbers are suppose to mean. Thus negative vectors are in opposite direction to positive vectors with zero vector, positive or negative, as unbiasedness towards positive or negative vectors; negative heights (depths) are in opposite direction to positive heights (altitudes) with positive zero representing no altitude and negative zero representing no depth; negative possessions (debts) are in opposite direction to positive possessions with positive zero representing no possession and negative zero indicating no debt.

3.4 The Zero as an Infinitely Small Quantity

Jean le Rond d'Alembert made one of the most famous assertions relating to infinitely small quantities:

> A quantity is something or nothing: if it is something, it has not yet vanished; if it is nothing, it has literally vanished. The supposition that there is an intermediate state between these two is a chimera.

In the past, many of the excellent writers were accustomed to commence their work on calculus with a presentation of the nature of infinitesimals as "ghosts of departed quantities", and then a contemplation of its arithmetic. The grand truth of the infinitesimals takes us to the absolute nothing. A right understanding of them, especially in their relation to the absolute nothing, is absolutely essential if we are to be preserved from fundamental error in the foundation of the calculus. If the foundation itself be faulty, then the building erected on it cannot be sound; and if we err in our conceptions of this basic element of analysis, then just in proportion as we do so will our grasp of the foundation of the calculus be inaccurate.

We are well aware that the high ground we are here treading is new and strange to almost all of our readers; for that reason it is well to move slowly.

Note well that the unit comparative nothing **0** is not only absolute nothing but also an infinitely small quantity, whereby it is implied an infinitely small quantity is equal to absolute nothing. This is so because **0** is the reciprocal of an infinitely large quantity. This will become evident as we proceed.

Let our appeal be to combinatorics. Consider the recursive formula of the factorial of $n + 1$:

$$(n + 1)! = (n + 1) \cdot n!$$

If we set $n = -1$, we get $\mathbf{0}! = \mathbf{0} \cdot (-1)!$ which we rewrite as

$$\mathbf{0} = \frac{\mathbf{0}!}{(-1)!}.$$

Taking $\mathbf{0}! = 1$ we write

$$\mathbf{0} = \frac{1}{(-1)!}.$$

At this stage, we turn to numerical method in order to show again that

$$\mathbf{0} = \frac{1}{(-1)!}.$$

The value of 0! is always taken to be, as a convention, unity. This fact may be reached by numerical analysis. For if we use the computer to compute the values of the factorials of $0.1, 0.01, 0.001, ...$, we shall obtain the data given in Table 1. The figures in the second column of this table approach unity as $n \to 0$. We may conclude from this that $0! = 1$.

An understanding of this fact prepares us for the assigning of a numerical value for the infinite $(-1)!$. We can continue to use our numerical method of reasoning. The starting point is the computation of the factorials of $-0.9, -0.99, -0.999, ...$ whose limit is $(-1)!$.The results from our computer are put in Table 2.

Table 1. Factorials of $0.1, 0.01, 0.001, ...$

n	$n!$
0.1	0.951350769866873183629...
0.01	0.994325851191506037135...
0.001	0.999423772484595466114...
0.0001	0.999942288323162419080...
$\vdots$	$\vdots$

Table 2. Factorials of $-0.9, -0.99, -0.999, \ldots$

n	$n!$
-0.9	$\frac{0.951350769866873183629...}{0.1}$
-0.99	$\frac{0.994325851191506037135...}{0.01}$
-0.999	$\frac{0.999423772484595466114...}{0.001}$
-0.9999	$\frac{0.999942288323162419080...}{0.0001}$
$\vdots$	$\vdots$

Comparing the entries of this table with those of Table 2, we observe that there is some relation among the numbers in the second columns of both tables. The following may, therefore, be concluded from our observations:

$$(-0.9)! = \frac{(0.1)!}{0.1}$$
$$(-0.99)! = \frac{(0.01)!}{0.01}$$
$$(-0.999)! = \frac{(0.001)!}{0.001}$$
$$(-0.9999)! = \frac{(0.0001)!}{0.0001}$$

and so forth. Our conclusion is that in the limit

$$(-1)! = \frac{\mathbf{0}!}{\mathbf{0}}$$

which, setting $0! = 1$, becomes

$$(-1)! = \frac{1}{\mathbf{0}}$$

which is the same as the required result.

Now,

$$(-1)! = (-1) \cdot (-2)! = (-1) \cdot (-2) \cdot (-3)! = \cdots = (-1) \cdot (-2) \cdot (-3) \cdots .$$

It follows immediately that

$$\mathbf{0} = \frac{1}{(-1) \cdot (-2) \cdot (-3) \cdots};$$

that is, $\mathbf{0}$ is the ratio of unity to the infinite product $(-1) \cdot (-2) \cdot (-3) \cdots$.

It ought to be pellucid and incontrovertible to all, that infinitely small quantities are genuinely zeros, that is absolute nothing. A strong proof of the fact that the infinitely small are zeros is the identity

$$\frac{1}{(-1)\cdot(-2)\cdot(-3)\cdots} = \mathbf{0}.$$

For though the fraction

$$\frac{1}{(-1)\cdot(-2)\cdot(-3)\cdots}$$

is a number infinitely small as it is the ratio of unity to an infinitely large quantity, the infinite product of all negative integers, we are forced somewhat to assert that the infinitely small quantity is equal to the zero **0**.

3.5 The Ratio of Zero to Itself

Thus far we have dwelt upon the infinitesimal side of the zero **0**; now we turn more directly to the ratio of **0** to itself, namely

$$\frac{\mathbf{0}}{\mathbf{0}}.$$

The zero 0 cannot be expressed as a ratio of two numbers and therefore unrelated to infinite quantities and hence infinitely small quantities. This is so because the mathematical expression which gives rise to it is not related to any other mathematical expression. For instance, the expression $x - x$ which equals 0 is not equal to any other expression.

On the other hand, the expression $x - c$ which gives rise to the zero **0** when we let $x = c$ [9], [15] is equal to and hence related to another expression, viz

$$x - c = \frac{(x-c)!}{(x-c-1)!}.$$

It, therefore, follows that

$$\frac{x-c}{x-c} = \frac{(x-c)!}{(x-c-1)!} \times \frac{(x-c-1)!}{(x-c)!}.$$

So if we set $x = c$, we get

$$\frac{c-c}{c-c} = \frac{(c-c)!}{(c-c-1)!} \times \frac{(c-c-1)!}{(c-c)!}$$

which becomes

$$\frac{\mathbf{0}}{\mathbf{0}} = \frac{\mathbf{0}!}{(\mathbf{0}-1)!} \times \frac{(\mathbf{0}-1)!}{\mathbf{0}!}$$

which, in its turn, becomes

$$\frac{\mathbf{0}}{\mathbf{0}} = \frac{1}{(-1)!} \times \frac{(-1)!}{1} = \frac{(-1)!}{(-1)!} = \frac{(-1)\cdot(-2)\cdot(-3)\cdots}{(-1)\cdot(-2)\cdot(-3)\cdots} = 1.$$

That the ratio of infinite products, each of which equals an infinite number, gives a finite value has long been recognized. For instance in 1665 Wallis published his famous infinite product for π:

$$\frac{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6\cdot 8\cdot 8\cdot 10\cdot 10\cdot 12\cdot 12\cdots}{1\cdot 3\cdot 3\cdot 5\cdot 5\cdot 7\cdot 7\cdot 9\cdot 9\cdot 11\cdot 11\cdot 13\cdots} = \frac{\pi}{2}$$

where the numerator

$$2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6\cdot 8\cdot 8\cdot 10\cdot 10\cdot 12\cdot 12\cdots$$

equals an infinite quantity and the denominator

$$1\cdot 3\cdot 3\cdot 5\cdot 5\cdot 7\cdot 7\cdot 9\cdot 9\cdot 11\cdot 11\cdot 13\cdots$$

equals another infinite quantity. In 1873 Catalan proved the Wallis-type identities

$$\frac{4\cdot 4\cdot 8\cdot 8\cdot 16\cdot 16\cdot 32\cdot 32\cdot 64\cdot 64\cdot 128\cdot 128\cdots}{3\cdot 5\cdot 7\cdot 9\cdot 11\cdot 13\cdot 15\cdot 17\cdot 19\cdot 21\cdot 23\cdot 25\cdots} = \frac{\pi}{2\sqrt{2}}$$

and
$$\frac{2\cdot 2\cdot 6\cdot 6\cdot 10\cdot 10\cdot 14\cdot 14\cdot 18\cdot 18\cdot 22\cdot 22\cdots}{1\cdot 3\cdot 5\cdot 7\cdot 9\cdot 11\cdot 13\cdot 15\cdot 17\cdot 19\cdot 21\cdot 23\cdot 25\cdots}=\sqrt{2}.$$
Number pattern is an important facet of mathematics. Consider the perfect flow of integer numbers from negative to positive:

$$\ldots,\quad -2,\quad -1,\quad \mathbf{0},\quad 1,\quad 2,\quad \ldots.$$

If we multiply each integer by its own reciprocal, we get the new sequence:

$$\ldots,\quad \frac{-2}{-2},\quad \frac{-1}{-1},\quad \frac{\mathbf{0}}{\mathbf{0}},\quad \frac{1}{1},\quad \frac{2}{2},\quad \ldots.$$

If there must be no break in this sequence, each term must equal to 1 without any exception. Thus $\mathbf{0}/\mathbf{0}=1$.

The orthodox notion of zero is that all zeros are equal to one another; for instance, $\mathbf{0}=2\times\mathbf{0}=3\times\mathbf{0}=\cdots$. This is not the behavior of zero. Mathematicians are only guessing this; they have no proof.

Just as $1,2,3,\ldots$ all have the same meaning of absolute something because they are real magnitudes or sizes, the zeros $\mathbf{0},2\times\mathbf{0},3\times\mathbf{0},\ldots$ all have the same meaning of absolute nothing because they are not magnitudes or sizes. But just as 1 and 2 are not equal analytically, even so $\mathbf{0}$ and $2\times\mathbf{0}$ are not equal analytically for while the zero $\mathbf{0}$ equals the infinitesimal

$$\frac{1}{(-1)\cdot(-2)\cdot(-3)\cdots}$$

the zero $2\times\mathbf{0}$ equals the infinitesimal

$$\frac{2}{(-1)\cdot(-2)\cdot(-3)\cdots}$$

which in turn equals
$$\frac{1}{(-3)\cdot(-4)\cdot(-5)\cdots}.$$
Thus $2\times\mathbf{0}/\mathbf{0}$ does not equal $\mathbf{0}/\mathbf{0}$. Rather $2\times\mathbf{0}/\mathbf{0}=2$; the two $\mathbf{0}$'s cancels out each other because they are analytically identical.

We have thus come to the following conclusion.

Theorem 3.1. *The division of zero* $\mathbf{0}$ *by itself is unity. That is*

$$\frac{\mathbf{0}}{\mathbf{0}}=1.$$

3.6 Principle of Impending Operation on Zero

3.6.1 Zero Paradox

The orthodox conception of the number zero, which in this paper is denoted $\mathbf{0}$, has often led to apparent proofs of false statements. Care must therefore be taken when carrying out calculations with zero. Failing to do so can result in proofs that two different numbers are equal to each other.

Start from
$$1=1.$$

Write this as
$$1\times\mathbf{0}=1\times\mathbf{0}.$$

Rewrite this as

$$\mathbf{0} = 1 \times \mathbf{0}.$$

Next rewrite this as

$$2 \times \mathbf{0} = 1 \times \mathbf{0}.$$

Now divide both sides by **0**. We get

$$2 = 1$$

which is incorrect. The error lies in the general view that $a \times \mathbf{0} = \mathbf{0}$ where a is a real number. Assuming

$$\frac{\mathbf{0}}{\mathbf{0}} = 1$$

so that it can be said that **0** subscribes to the general theorem that $x/x = 1$, it can be seen, from the orthodox axiom $a \times \mathbf{0} = \mathbf{0}$, that

$$a = 1$$

which is a contradiction.

Let a, b, c be three different numbers. If

$$a \times b = c$$

then c can be factored into a and b. In modern mathematics this is not true when either a or b or both are equal to **0**. For instance, if $b = \mathbf{0}$, then $c = \mathbf{0}$. How can we obtain the number a by factoring $c = \mathbf{0}$? This is impossible since zero **0** is inappropriately taken to be the product of **0** and any number whatsoever.

3.6.2 Principle of Impending Operation on Imaginary Unit

Whenever a number, expected to conform to an axiom or theorem which other numbers subscribe to, behaves in such a way that its use with the axiom or theorem leads to contradiction, fallacy, paradox, or indeterminacy, a law is usually enacted to guild its use. For instance, an improper use of the number $\sqrt{-1}$ for the imaginary unit i leads to the apparent proof of a false statement. If we work with the generally accepted theorem

$$\sqrt{a} \times \sqrt{b} = \sqrt{a \times b}$$

in the usual manner, we have

$$i \times i = \sqrt{-1} \times \sqrt{-1} = \sqrt{(-1) \times (-1)} = \sqrt{1} = 1.$$

That is

$$i^2 = 1.$$

But we know that

$$i^2 = -1.$$

This is, therefore, a contradiction.

It is not that the symbol $\sqrt{}$ does not have the clear-cut meaning with complex numbers that it has with real numbers. It is true that

$$(-1) \times (-1) = (-1)^2 = 1$$

but if there remains a calculation involving the power 1/2, we must retain $(-1)^2$ and use it instead of 1 in this calculation. Thus

$$\sqrt{(-1)} \times \sqrt{(-1)} = \sqrt{(-1)(-1)} = \left[(-1)^2\right]^{\frac{1}{2}} = -1.$$

3.6.3 Bhaskara's Principle of Impending Operation on Zero

We set the stage with the following extract from *Nothing that is* by Robert Kaplan [11]:

> But if you really would divide by zero, then all numbers would be the same. Why? Our Indian mathematicians help us here: any number times zero is zero-so that $6 \cdot 0 = 0$ and $17 \cdot 0 = 0$. Hence, $6 \cdot 0 = 17 \cdot 0$. If you could divide by 0, you'd get
>
> $$\frac{6 \cdot 0}{0} = \frac{17 \cdot 0}{0},$$
>
> the zeros would cancel out and 6 would equal 7. They aren't equal, so you can't legitimately divide by 0: $a/0$ doesn't mean anything.

Bhaskara II, whose notion of zero divided by zero has been heavily criticized worldwide [16], [12], was a man of great sagacity, both for understanding all things and persuading his readers. He was not mistaken in his opinion. For this reason he began to have higher notions of mathematics and astronomy than others had, and he determined to renew and to change the opinion all happened to have concerning multiples of 0 and division by 0, for he was the first that ventured to publish this notion [17], [18]:

> When a number (a) is multiplied by cipher (0), the product ($a \cdot 0$) is cipher (0); but in case any operation remains to be done, cipher (0) is considered to be the multiplier (i.e. the expression $a \cdot 0$ is to be retained), and if cipher (0) also becomes the divisor (in the remaining operation), the number (a) is considered unchanged (i.e. $a \cdot 0/0 = a$ in the remaining operation where 0 is a divisor).

Let us now help Kaplan here. Bhaskara says that

$$a \cdot \mathbf{0} = 0.$$

Suppose, after the operation $a \cdot \mathbf{0}$, there is an approaching operation in which $\mathbf{0}$ is a divisor. According to Bhaskara, we should use the form $a \cdot \mathbf{0}$ (and not merely 0 to which $a \cdot \mathbf{0}$ equals) in this new operation such that when it is divided by $\mathbf{0}$, the result gives a, that is

$$\frac{a \cdot \mathbf{0}}{\mathbf{0}} = a.$$

The above equation is often called Bhaskara's identity.

Here is a simple demonstration of the above point. If we set $x = 1$ in the expression

$$\frac{x^2 - 1}{x - 1},$$

we get

$$\frac{1^2 - 1}{1 - 1}$$

which is written as

$$\frac{0}{0}$$

and termed *indeterminate*. The mystery is easily resolved with a little algebra:

$$\frac{x^2 - 1}{x - 1} = \frac{(x + 1)(x - 1)}{x - 1}.$$

Setting $x = 1$ gives

$$\frac{1^2 - 1}{1 - 1} = \frac{(1 + 1)(1 - 1)}{1 - 1} = \frac{2 \cdot \mathbf{0}}{\mathbf{0}} = 2.$$

Table 3. Table of values

x	$x-1$	x^2-1
1.1	0.1	2.1×0.1
1.01	0.01	2.01×0.01
1.001	0.001	2.001×0.001
$\vdots$	$\vdots$	$\vdots$

The evaluation of

$$\frac{x^2-1}{x-1}$$

at $x=1$ is so obvious as not to need technical definition. If x is nearly 1, then $x-1$ is nearly $\mathbf{0}$ and x^2-1 is nearly $2\cdot\mathbf{0}$ as can be seen in the Table 3. Eventually, when $x=1$, $x-1=\mathbf{0}$ and $x^2-1=2\cdot\mathbf{0}$. This simple example shows that Bhaskara's identity holds true.

3.6.4 Improved Principle of Impending Operation on Zero

Here, we improve the principle of impending operation on zero: "If in some mathematical calculations, the zero $\mathbf{0}$ is likely to occur frequently, then, though $a\cdot\mathbf{0}^n=0$ where a is a finite number and n is any positive number, one should maintain the form $a\cdot\mathbf{0}^n$ in the rest of the operations until the final operation with $\mathbf{0}$ is reached. This is because if a finite number is multiplied by zero and divided by the same zero, then the result is the finite number".

4 On Infinities

4.1 Infinities as Zero Divisors

Expressions of division of finite numbers by zeros are forced upon our notice in very many unrealistic problems such as the famous Courier Problem which shall be treated later. It is proper to call the simplest case $\frac{1}{0}$ the **unit infinity** because of the dividend being unity and the divisor being equal the unit zero. This infinity does not correspond to any real quantity whatever. But, remembering the restricted nature of real quantity, it is possible that such an expression is unreal only in the sense that the things it describes cannot be obtained in nature. If the restriction can be removed by an extension of our conception beyond reality, the expression may, perhaps, become as real to us as complex numbers now are.

While the unit infinity is not a real number, we may give to it the definition:

Definition 4.1. The unit infinity is the symbol of value $\frac{1}{\mathbf{0}}$ which is consistent with this one condition:

$$\mathbf{0}\times\frac{1}{\mathbf{0}}=\frac{\mathbf{0}}{\mathbf{0}}=1.$$

As the fraction $\frac{1}{\mathbf{0}}$ may be defined as such a symbol which multiplies $\mathbf{0}$ to produce unity, it may be termed the reciprocal of $\mathbf{0}$.

The notion of infinity is a difficult idea, and this in three respects. First, in the understanding of it. Unless we give ourselves to the study of the true nature of zero, great pains and diligence are called for in the searching for the nature of infinity. Second, in the acceptation of it. This presents a much greater difficulty, for when the intuition perceives what some mathematical processes reveal thereon, the mind is loath to receive such clear truth. Third, in the explanation of it. No novice is competent to present this subject in its mathematical perspective.

4.2 Infinite Gideon's Bibles Demanded

The idea of infinity arises in the following dialogue among three sisters, Ochuko, Ese and Onajite.

Ochuko: How can the notion of infinity arise in real life?

Ese: That is something I find so difficult to comprehend.

Ochuko: Let's begin with ordinary numbers because these are what our finite minds can conceive without any question popping up. Without any difficulty, we understand these definite numbers when they are used to represent magnitudes of quantities.

Onajite: What do you mean by magnitudes of quantities? Throw more light on these terms– magnitude and quantity.

Ochuko: By quantity I mean amount; how many or how much there is of something. If there are five bibles in a shelve, we use the number 5 to represent the quantity of bibles in the shelve.

Onajite: Alright, I now understand what you mean by quantity. What about the term magnitude.

Ochuko: Magnitude simply means size. How large a quantity or amount is from nothing implies magnitude.

Onajite: Alright, I am done.

Ochuko: Let's begin our talk on infinity with the ratio of ordinary numbers. Take the fraction

$$\frac{12}{3}.$$

What is its quotient? To get the quotient, we ask ourselves the question: How many bibles can I buy from a seller with my 12 naira if each bible costs 3 naira? It is easily seen that my money will get for me four bibles. If, however, the seller gives me only one bible, I will surely ask for the rest three because what I pay, the 12 naira, is more than the cost of one bible by 9 naira. Suppose the seller gives me another bible. I will ask for the remaining two.

Ese: Suppose he says that it is balanced.

Ochuko: I will tell him no because the 12 naira is more than the cost of two bibles by 6 naira. If at last he gives me two additional bibles, I will tell him it is balanced because the 12 naira is now equal to the total cost of the four bibles he has given me.

Ese: So the quotient of the fraction you propose is 4.

Ochuko: You get it.

Ese: I have no trouble with understanding this. Can this explanation of yours be extended to cases where zero is a divisor? Take, for instance, the fraction

$$\frac{200}{0}.$$

[The non-boldfaced 0 here and in other places represents any zero] I wish to see how you explain this.

Ochuko: Listen to this. A seller is supplied hundred Gideon bibles, each stamped with the statement "not to be sold". He is instructed to distribute them to his buyers free of charge so that the Gospel of Christ can reach as many as possible.

A pastor hears of this and wants to collect all the bibles from the seller. He knows the seller will never release them to him so he decides that he will pay the seller some money to give them to him. The pastor meets the seller, puts his hand in his pocket, brings out 200 naira, and gives it to him.

"Give me Gideon bibles", says the pastor. The seller, being greedy, collects the money and gives the pastor only one of the bibles. The pastor says, "It is not balanced." "How?", the seller replies. The pastor explains that what he pays is more than the selling price of the bible by 200 naira. The seller laughs and gives him nine more bibles. The pastor tells him that the 200 naira he pays is still more than the total price of the ten bibles by 200 naira. "What kind of thing this is!", the seller cries. Then he continues, "I know what he wants." The seller dashes into one of his rooms, brings out the rest of the bibles and gives them to the pastor.

The pastor, thinking that the seller is still having more of the bibles, argues further that the total price of the 100 bibles is zero and that his 200 naira exceeds it by 200 naira. The seller then reasons and says, "Pastor, there is no definite number of the Gideon bibles I will give you such that their total cost balances the 200 naira you pay. I mean *there is* **no definite number** *of Gideon bibles I will give you, the total cost cannot balance the 200 naira you pay since each bible costs nothing as it is free of charge and not to be sold*."

Ese: Splendid explanation! I can tell the answer from this story. The number of the free Gideon bibles the seller will give the pastor must be without end.

Ochuko: This is the concept of infinity. Infinity is the number representing something without any bound. In the case proposed, the fraction

$$\frac{200}{0}$$

is an infinite (endless) number.

4.3 Unit Infinity as a Limit

It is well known that the limit of the sequence of the multiplicative inverses of the natural numbers

$$1, \quad \frac{1}{2}, \quad \frac{1}{3}, \quad \frac{1}{4}, \quad \ldots$$

is the number **0**. In as far as this is true, one is inevitably by way of correspondence as in

$$\begin{gathered} \frac{1}{1} \leftrightarrow 1 \\ \frac{1}{2} \leftrightarrow 2 \\ \frac{1}{3} \leftrightarrow 3 \\ \frac{1}{4} \leftrightarrow 4 \\ \vdots \end{gathered}$$

to suppose that the limit of the sequence of the natural numbers

$$1, \quad 2, \quad 3, \quad 4, \ldots$$

is the unit infinity

$$\boldsymbol{\infty}.$$

These limits, $\mathbf{0}$ and $\boldsymbol{\infty}$, can be placed respectively at the two extremities of the sequence of numbers with whole numbers and their multiplicative inverses extending endlessly in opposite directions:

$$\mathbf{0}, \quad \ldots, \quad \frac{1}{3}, \quad \frac{1}{2}, \quad 1, \quad 2, \quad 3, \quad \ldots, \quad \boldsymbol{\infty}.$$

Any two numbers in the sequence that are equidistant from the number 1 are multiplicative inverses of each other, thus producing 1 when multiplied together:

$$n \cdot \frac{1}{n} = 1,$$

such that for the two limits or extremities or final terms, $\mathbf{0}$ and $\boldsymbol{\infty}$, one could have as well

$$\mathbf{0} \cdot \frac{1}{\mathbf{0}} = 1$$

or simply

$$\mathbf{0} \cdot \boldsymbol{\infty} = 1.$$

The fact that the unending sequence of natural numbers has a real end can be explained by the well known iterative process of inscribing in a circle, one after the other, the infinite sequence of regular polygons. The two figures, the polygon and the circle, come to resemble each other more and more as the number of vertices of the polygon increases. Each polygon in the sequence has a successor and therefore there is the potential of extending the sequence again and again without coming to a stop. See the fig. 2.

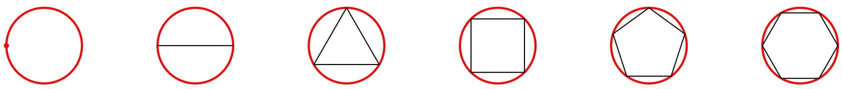

Fig. 2. Regular polygons inscribing a circle

Nunez in [19] throws some light on this iterative process:

> Thinking in terms of actual infinity imposes an end at infinity where the entire infinite sequence does have a final resultant state, namely a circle conceived as a regular polygon with an infinite number of sides (see Fig. 2). This circle has all the prototypical properties circles have (i.e., area, perimeter, a center equidistant to all points on the circle, π being the ratio between the perimeter and the diameter, etc.) but conceptually it is a polygon.

Thus, the sequence of regular polygons is endless but it is conceived as being completed. The final resultant state is a circle conceived as a polygon with endless number of vertices. It turns out that the sequence of numbers of vertices of the regular polygons, though endless, must be conceived, by way of correlation, as being completed. To make this point even clearer we proceed to consider the theory of infinite set associated with Georg Cantor. Let set A consists of regular polygons with the same circumscribing circle,viz

$$\mathbf{A} = \{\text{monogon, digon, trigon, tetragon, penagon}, \ldots\}$$

and set $\mathbf{B}$ the numbers of vertices of the polygons in A, viz

$$\mathbf{B} = \{1, 2, 3, 4, 5, \ldots\}$$

which is the set of natural numbers, the cardinal number of which is usually denoted $\aleph_0$. These two sets can be lined up one-to-one as shown in the scheme below.

$$\begin{aligned} \text{Set } \mathbf{A} &\leftrightarrow \text{Set } \mathbf{B} \\ \text{monogon} &\leftrightarrow 1 \\ \text{digon} &\leftrightarrow 2 \\ \text{trigon} &\leftrightarrow 3 \\ \text{tetragon} &\leftrightarrow 4 \\ \text{pentagon} &\leftrightarrow 5 \end{aligned}$$

and so on. To each polygon in set **A** there thus corresponds a natural number in set **B** such that the two sets are equally unending, and in exactly the same way. Hence, both sets have the same cardinal number, namely, $\aleph_0$. Since the circle is the final element of set **A**, set **B** must possess a final element corresponding to the circle in **A** and is the number of vertices of the circle, and hence the final natural number which is $\aleph_0$ or as we have shown 1/0.

4.4 Signs of Infinity

For further comprehension of infinity, it is necessary to inquire into its signs and an important starting point is the discourse on the signs of zero. If we multiply each term of the sequence of integers

$$\cdots, -3, \quad -2, \quad -1, \quad \mathbf{0}, \quad 1, \quad 2, \quad 3, \cdots$$

by -1, we have the inverted sequence of integers

$$\cdots, +3, \quad +2, \quad +1, \quad -\mathbf{0}, \quad -1, \quad -2, \quad -3, \cdots.$$

As we compare the two sequences above, we observe **0** and its additive inverse $-\mathbf{0}$. These zeros, **0** and $-\mathbf{0}$, are situated at the centre of both sequences, and while **0** is the transition integer through which integers pass from positive to negative, $-\mathbf{0}$ is the transition integer through which integers pass from negative to positive. They, therefore, coincide with each other.

If we take the convention of always passing from the right side to the left side of the origin 0 of an x-axis, then we observe that immediately we reach the starting point of all positive integers, the number $x = +\mathbf{0}$, we have as well reached the starting point of all negative integers, the number $x = -\mathbf{0}$, so that $+\mathbf{0}$ is not greater or less than $-\mathbf{0}$; they coincide at the same point, the origin 0. They are, therefore, equal to each other and both representing absolute nothing. Thus in the integer number line, the origin is a point where an integer is characterized as having both positive and negative signs and this integer is the zero **0**. We may say that the zero **0** comes in the pair $(+\mathbf{0}, -\mathbf{0})$.

Like zero, infinity comes in pairs, positive and negative. If we use the familiar sign ∞ to denote $1/\mathbf{0}$, then we get the following relations

$$\frac{1}{+\mathbf{0}} = +\infty$$

and

$$\frac{1}{-\mathbf{0}} = -\infty.$$

In integer number line these infinities may be placed at the extremities of the line with the positive infinity $+\infty$ at the right extreme and negative infinity $-\infty$ at the left extreme.

In such a line $+\infty$ is the positive infinity associated with $+\mathbf{0}$ and $-\infty$ is the negative infinity associated with $-\mathbf{0}$. But because both $+\mathbf{0}$ and $-\mathbf{0}$ coincide at the same single point, the centre and origin 0 which is the dividing line between all positive and all negative integers, the positive infinity $+\infty$ and the negative infinity $-\infty$ coincide and are reached at the same time, though both are at equal unending distances away from 0 and in opposite directions.

Consider the function

$$f(x) = \frac{x}{x-2} = \frac{2}{x-2} + 1,$$

the graph of which is shown in Fig. 3.

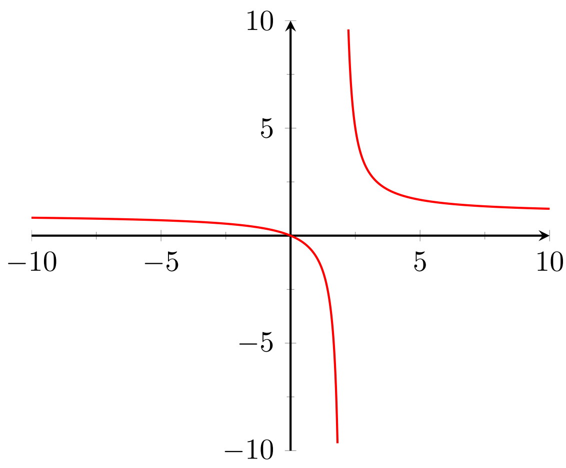

Fig. 3. Graph of $f(x) = \frac{x}{x-2}$

To study the behaviour of the curve, let us move gradually along the $x-$axis from right to left. We notice that as we approach $x = 2$, the curve approaches the vertical asymptote $x = 2$ and the function value which has been positive and finite approaches a positive and infinite number. When we reach $x = 2$, the function value equals the positive infinite number $2/0 + 1$, the quantity beyond which the function value will never go. At this stage, starting from any finite x, the curve is said to have extended without bound. Since, as we already noted, the zero number $+0$ coincides with the zero number -0, it turns out that the function value reaches the negative infinite number $-2/0+1$ when it has reached the positive infinite number $2/0 + 1$ at $x = 2$. Hence, as we move away from $x = 2$, the curve returns from the negative infinite number already mentioned and the function value becomes negative and finite. As we move farther and farther away from $x = 2$, the curve bends more and more away from the vertical asymptote $x = 2$.

5 Infringement of Mathematical Provisos

The most eminent illustration of the subject of zero and infinity we have been treating is in the infringement of the provisos in certain mathematical formulas. Mathematicians (not Mathematics) hate division by zero because they fail to comprehend the true behaviours of zero and infinity. For this reason, whenever they come up with a formula, they guild it from division by zero, placing a condition for which the formula is inapplicable. In this section, we shall employ the knowledge we have acquired of zero and infinity to infringe upon the mathematical provisos placed on certain formulas.

5.1 Provisos in Series

5.1.1 Infinite Sum of the Alternating Harmonic Series

In one of his works, Euler showed that

$$1^n - 2^n + 3^n - 4^n + \cdots = (-1)^s \frac{2^{n+1}-1}{n+1} B_{n+1}, \quad n \neq -1$$

where $s = [(n+1)/2]$ is the integer part of $(n+1)/2$ and B_n is the nth Bernoulli number. If we disobey the proviso $n \neq -1$ and set $n = -1$, we get

$$1^{-1} - 2^{-1} + 3^{-1} - 4^{-1} + \cdots = (-1)^{[(-1+1)/2]} \frac{2^{-1+1}-1}{-1+1} B_{-1+1}$$

which simplifies to

$$\frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = (-1)^{[(\mathbf{0})/2]} \frac{2^{\mathbf{0}} - 1^{\mathbf{0}}}{\mathbf{0}} B_{\mathbf{0}}.$$

The identity we shall use here and hereafter is

$$\frac{b^{\mathbf{0}} - a^{\mathbf{0}}}{\mathbf{0}} = \ln b - \ln a. \tag{5.1}$$

To prove this identity, we evaluate

$$\frac{b^x - a^x}{x}$$

at $x = \mathbf{0}$. This is done by first taking the series expansion of a^x:

$$a^x = 1 + x \ln a + \frac{(x \ln a)^2}{2!} + \cdots$$

Therefore the difference $b^x - a^x$ is expressed as:

$$b^x - a^x = 1 - 1 + (\ln b - \ln a)\, x + \frac{\left(\ln^2 b - \ln^2 a\right) x^2}{2!} + \cdots$$

which, omitting $1 - 1$ as it is irreversibly equal to 0, becomes

$$b^x - a^x = (\ln b - \ln a)\, x + \frac{\left(\ln^2 b - \ln^2 a\right) x^2}{2!} + \cdots .$$

Letting $x = \mathbf{0}$ gives

$$b^{\mathbf{0}} - a^{\mathbf{0}} = (\ln b - \ln a)\, \mathbf{0} + \frac{\left(\ln^2 b - \ln^2 a\right) \mathbf{0}^2}{2!} + \cdots .$$

Dividing both sides by $\mathbf{0}$ furnishes

$$\frac{b^{\mathbf{0}} - a^{\mathbf{0}}}{\mathbf{0}} = (\ln b - \ln a) + \frac{\left(\ln^2 b - \ln^2 a\right) \mathbf{0}}{2!} + \cdots$$

which becomes the final result

$$\frac{b^{\mathbf{0}} - a^{\mathbf{0}}}{\mathbf{0}} = \ln b - \ln a \tag{5.2}$$

since

$$\frac{\left(\ln^2 b - \ln^2 a\right) \mathbf{0}}{2!} + \cdots$$

is equal to naught.

Based on this we have

$$\frac{2^{\mathbf{0}} - 1^{\mathbf{0}}}{\mathbf{0}} = \ln 2 - \ln 1 = \ln 2.$$

and with the understanding that $B_{\mathbf{0}} = 1$, we write

$$\frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \ln 2.$$

5.1.2 Infinite Sum of the Harmonic Series

Let us consider the employment of the identity $(-1)! = 1/\mathbf{0}$ in assigning a sum to the divergent harmonic series defined as

$$\sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots .$$

We start with the fundamental property of $(-1)!$: $(-1)! = 1/\mathbf{0}$. We then find the natural logarithm of both sides of the equation and obtain the relation $\ln(-1)! = \ln(1/\mathbf{0})$ which, on being rearranged, gives $\ln(-1)! = -\ln \mathbf{0}$. With the help of this relation we will show that the sum of the divergent harmonic series $\sum_{k=1}^{\infty} 1/k$ is $\ln(-1)!$, an infinitely large number less than $(-1)!$. Our aim at this point is, therefore, to demonstrate that $\ln(-1)!$ is the sum of the harmonic series, that is to say,

$$\sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots = \ln(-1)!. \tag{5.3}$$

The possibility of such a relation as (5.3) is suggested by inspecting the Taylor series expansion of $\ln(x + 1)$ [20],

$$\ln(x + 1) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots ,$$

and letting $x = -1$. Accomplishing these, we obtain the following:

$$\ln \mathbf{0} = -\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots\right)$$

$$-\ln \mathbf{0} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots .$$

Employing $\ln(-1)! = -\ln \mathbf{0}$, we arrived at the required result

$$\ln(-1)! = 1 + \frac{1}{2} + \frac{1}{3} + \cdots = \sum_{k=1}^{\infty} \frac{1}{k}.$$

Let us now turn to the derivation of a famous formula in analysis in order to give the reader an idea of the flavor of $\ln(-1)!$. There is a very interesting formula discovered by Euler in his 1776 paper [21], [15], which presents a beautiful means of computing Euler's constant γ. This formula is

$$1 - \gamma = \sum_{k=2}^{\infty} \frac{\zeta(k) - 1}{k}.$$

We now proceed to derive this formula and we begin from the Maclaurin series expansion for $\ln(x)!$ which reads

$$\ln(x!) = -\gamma x + \sum_{k=2}^{\infty} (-1)^k \frac{\zeta(k)}{k} x^k, \quad x \neq -1.$$

If we ignore the condition $x \neq -1$, letting $x = -1$, we obtain the result

$$\ln(-1)! = \gamma + \sum_{k=2}^{\infty} \frac{\zeta(k)}{k}.$$

We have already established that the natural logarithm of $(-1)!$ is the sum of the harmonic series $\sum_{k=1}^{\infty} \frac{1}{k}$. If we then replace $\ln(-1)!$ with the sum $\sum_{k=1}^{\infty} \frac{1}{k}$, we procure for ourselves

$$\sum_{k=1}^{\infty} \frac{1}{k} = \gamma + \sum_{k=2}^{\infty} \frac{\zeta(k)}{k}$$

which results in

$$1+\sum_{k=2}^{\infty}\frac{1}{k}-\sum_{k=2}^{\infty}\frac{\zeta(k)}{k}=\gamma$$

which in turn furnishes our required formula

$$1+\sum_{k=2}^{\infty}\frac{1-\zeta(k)}{k}=\gamma$$

or

$$1-\gamma=\sum_{k=2}^{\infty}\frac{\zeta(k)-1}{k}.$$

5.1.3 Euler's Constant

Here we use the way of zero and infinity to derive the famous identity

$$\gamma = H_\Omega - \ln\Omega$$

where Ω is the infinite integer for which $H_\Omega = 1+1/2+1/3+\cdots+1/\Omega$ is the harmonic series and $\gamma = 0.57721\ldots$ is the Euler's constant. We start with the sums of powers formula

$$\sum_{k=1}^{n}k^m=\frac{(B+n)^{m+1}-B^{m+1}}{m+1}\quad m\neg-1$$

where B^m equals the mth Bernoulli number B_m. If we break the proviso $m\neg-1$ by setting $m=-1$, we get

$$\sum_{k=1}^{n}k^{-1}=\frac{(B+n)^{-1+1}-B^{-1+1}}{-1+1}$$

which becomes

$$\sum_{k=1}^{n}k^{-1}=\frac{(B+n)^{\mathbf{0}}-B^{\mathbf{0}}}{\mathbf{0}}.$$

If we consider (5.2), the above result becomes

$$\sum_{k=1}^{n}\frac{1}{k}=\ln(B+n)-\ln B=\ln\left(\frac{B+n}{B}\right)=\ln\left(1+\frac{n}{B}\right).$$

This, setting $\sum_{k=1}^{n}\frac{1}{k}=H_n$ where H_n is the nth harmonic number, becomes

$$H_n=\ln\left(\frac{B+n}{B}\right)=\ln\left(1+\frac{n}{B}\right). \tag{5.4}$$

The question now is, What is B? To answer this question we need to express B in terms of n and set $n=1,2,3,\ldots$ to see what would happen to B. Now B expressed as the subject is

$$B=\frac{n}{e^{H_n}-1}$$

where e is Euler's number. Computing B for the first few values of n, we observe that B varies with n. We conclude that B is a variable depending on n. Let now B be rewritten as the function $B(n)$. The above equation becomes

$$B(n)=\frac{n}{e^{H_n}-1}.$$

Table 4. Values of $B(n)$

n	H_n	$B(n)$
10	2.92896 . . .	0.62117 . . .
100	5.18737 . . .	0.56742 . . .
1000	7.48547 . . .	0.56205 . . .
10000	9.78760 . . .	0.56151 . . .
100000	12.0901 . . .	0.56146 . . .
1000000	14.3927 . . .	0.56146 . . .
$\vdots$	$\vdots$	$\vdots$

It remains to compute the functional value of $B(n)$ when H_n becomes the harmonic series $H_\Omega = 1 + 1/2 + 1/3 + \cdots + 1/\Omega$. To perform this we construct a table of values of $B(n)$ as n becomes larger and larger. From this table, we see that as n becomes larger and larger, $B(n)$ becomes closer and closer to $0.561459\ldots = e^{-\gamma}$ where $e = 2.71828\ldots$. Let Ω be the value of n for which H_Ω is the harmonic series. It follows that

$$B(\Omega) = 0.561459\ldots = e^{-\gamma}.$$

We see immediately that, setting $n = \Omega$ in (5.4), the harmonic number becomes

$$H_\Omega = \ln\left(1 + \frac{\Omega}{e^{-\gamma}}\right)$$

which becomes

$$H_\Omega = \ln\left(1 + \Omega e^{\gamma}\right).$$

Since Ω is an infinite quantity, it is immutable in the presence of finite addends or minuends. Thus the above equation becomes

$$H_\Omega = \ln\left(\Omega e^{\gamma}\right)$$

which becomes the required identity

$$\gamma = H_\Omega - \ln\Omega.$$

5.2 Provisos in Calculus

We start with the integral

$$\int \frac{(\ln ax)^n}{x}dx = \frac{(\ln ax)^{n+1}}{n+1}, \quad n \neq -1.$$

If we violate the strict proviso $n \neq -1$ by setting $n = -1$ in

$$\int_A^B \frac{(\ln ax)^n}{x}dx = \left[\frac{(\ln ax)^{n+1}}{n+1}\right]_A^B,$$

we get

$$\int_A^B \frac{(\ln ax)^{-1}}{x}dx = \left[\frac{(\ln ax)^{\mathbf{0}}}{\mathbf{0}}\right]_A^B$$

which becomes

$$\int_A^B \frac{(\ln ax)^{-1}}{x}dx = \left[\frac{(\ln aB)^{\mathbf{0}}}{\mathbf{0}}\right] - \left[\frac{(\ln aA)^{\mathbf{0}}}{\mathbf{0}}\right]$$

which in its turn becomes

$$\int_A^B \frac{(\ln ax)^{-1}}{x} dx = \frac{(\ln aB)^{\mathbf{0}} - (\ln aA)^{\mathbf{0}}}{\mathbf{0}}.$$

This, on applying, becomes

$$\int_A^B \frac{(\ln ax)^{-1}}{x} dx = \ln(\ln aB) - \ln(\ln aA)$$

which becomes

$$\int_A^B \frac{(\ln ax)^{-1}}{x} dx = [\ln(\ln ax)]_A^B .$$

Thus we have the result

$$\int \frac{1}{x(\ln ax)} dx = \ln(\ln ax).$$

6 Mechanics

The practical illustrations of what we said about zeros and infinities are found in the various facet of Physics, the discipline over which most of mathematical findings have their triumph.

6.1 Velocity of Falling Bodies

Without the ratio of zeros it is impossible to obtain a clear notion of what is the velocity of a falling body at any instant of time. The mathematical discipline that treats of the method of obtaining this velocity is called today the Differential and Integral Calculus, but it was first called the Infinitesimal calculus by Leibniz or Fluxional Calculus by Newton. The average velocity of a body in any interval of time is defined as the ratio of the distance travelled to this time interval. If this time interval is made to approach zero as the limit, the ratio, consequently, approaches a limiting value. The limit which this ratio approaches as the time interval approaches zero is called the velocity of the body at the instant. This instant is zero as zero is the limit of the time interval.

Let us apply this method to bodies falling under the influence of gravity. We know from experiment that the distance fallen starting from rest is

$$s = \frac{1}{2}gt^2$$

where t is the time of falling and g is the acceleration of free fall. Suppose the time is increased by a small increment Δt. The distance fallen will increase by Δs obtained as follows. We start by writing

$$s + \Delta s = \frac{1}{2}g(t + \Delta t)^2.$$

This gives

$$s + \Delta s = \frac{1}{2}gt^2 + gt\Delta t + \frac{1}{2}g(\Delta t)^2.$$

But,

$$s = \frac{1}{2}gt^2.$$

Subtracting, we have

$$\Delta s = gt\Delta t + \frac{1}{2}g(\Delta t)^2.$$

Let $\Delta t = \mathbf{0}$. This gives us

$$\Delta s = gt \cdot \mathbf{0} + \frac{1}{2}g \cdot \mathbf{0}^2.$$

We wish to find the ratio of Δs to Δt at $\Delta t = \mathbf{0}$ for it is this ratio that furnishes the precise velocity of the falling body at any instant. We know that both Δt and Δs are absolute nothing, but because further operation of division by absolute nothing impends, we use the original forms of expressions $\mathbf{0}$ and $gt \cdot \mathbf{0} + \frac{1}{2}g \cdot \mathbf{0}^2$ for Δt and Δs respectively. Thus we obtain the velocity of the falling body at any instant as

$$\frac{\Delta s}{\Delta t} = \frac{gt \cdot \mathbf{0} + \frac{1}{2}g \cdot \mathbf{0}^2}{\mathbf{0}}$$

which, rewriting the ratio $\frac{\Delta s}{\Delta t}$ in the usual form $\frac{ds}{dt}$, becomes

$$\frac{ds}{dt} = \frac{gt \cdot \mathbf{0}}{\mathbf{0}} + \frac{1}{2}g\frac{\mathbf{0}^2}{\mathbf{0}} = gt + \frac{1}{2}g \cdot \mathbf{0}.$$

Since no further operation of division by nothing is expected, we write

$$\frac{ds}{dt} = gt + 0 = gt.$$

Thus the velocity of a falling body at any instant is gt.

6.2 Torricelli's Law

A tank holds water of volume V, which drains from a leak at the bottom, causing the tank to empty at time T. The tank drain faster when it is nearly full because the pressure on the leak is greater. Torricelli's Law gives the volume of the water remaining in the tank after time t as

$$V(t) = V\left(1 - \frac{t}{T}\right)^2.$$

Suppose the tank has no leak. The tank does not drain and, therefore, **no matter how large the elapsed time** *the tank will never be empty*. This implies $T = \frac{1}{\mathbf{0}}$ unit time. Thus the volume of the water in the tank after any given time t is

$$V(t) = V\left(1 - \frac{t}{\frac{1}{\mathbf{0}}}\right)^2 = V\left(1 - t \cdot \mathbf{0}\right)^2 = V.$$

From this it follows that the volume of water in the tank remains equal to the initial volume of the water in the tank at any time t.

6.3 Clairaut's Problem of the Couriers

Two couriers, A and B, travel in the same direction, CD, at the rates m and n miles an hour, respectively. If at any time, say 12 O'clock, A is at P, and B is at Q, a miles from P, at what time and at what place are they together?

Let t be the number of hours traveled, after 12 O'clock, to the place where A overtakes B, and d the number of miles travelled by A in t hours ; or the number of miles from P to the place where A overtakes B. Since the number of miles travelled by each, after 12 O'clock, equals the rate multiplied by the number of hours, we have

$$d = mt \quad \text{and} \quad d - a = nt.$$

Solving these equations, we have

$$t = \frac{a}{m-n} \quad \text{and} \quad d = \frac{am}{m-n} = \frac{an}{m-n} + a. \tag{6.1}$$

We will now examine these values under the condition $m = n$, that is when the couriers travel at the same rate. If we set $m = n$ in the equations (6.1), we get

$$t = \frac{a}{\mathbf{0}} \quad \text{and} \quad d = \frac{an}{\mathbf{0}} + a.$$

Since t is a fraction with a finite numerator and a zero denominator, the time it takes A to overtake B is endless, that is infinite. But endless time has no endpoint. The implication of this is that **no matter how large the time spent**, *A and B will never be together*. Similarly, the distance d, starting from P, A must cover in order to overtake B is endless since d is infinite. But endless distance has no endpoint. It follows then that **no matter how large the distance covered**, *A will never overtake B*. In essence there is no specific time when and no specific distance where A and B are together.

6.4 Work Done by a Force

Suppose that a force F pulls an object a finite distance s along a horizontal floor at an angle θ to the horizontal. The finite work done W by the force is related to the force by

$$F = \frac{W}{s\cos\theta}.$$

Let W and s remain finite and fixed. As θ is increased from zero, the force F is increased. When $\theta = 90°$, F becomes

$$F = \frac{W}{s\cos\left(\frac{\pi}{2}\right)} = \frac{W}{s\sin\mathbf{0}} = \frac{W}{s\left(\mathbf{0} - \frac{\mathbf{0}^3}{3!} + \frac{\mathbf{0}^5}{5!}\cdots\right)} = \frac{W}{s\cdot\mathbf{0}\left(1 - \frac{\mathbf{0}^2}{3!} + \frac{\mathbf{0}^4}{5!}\cdots\right)}.$$

Since no further operation involving zero impends, we simply write

$$F = \frac{W}{s\cdot\mathbf{0}}.$$

Thus the force F is endless or transfinite when $\theta = 90°$. It, therefore, follows that **no matter how large the force** F **applied**, *it can never perform any finite work over a given finite distance when it is inclined at* $90°$ *to the horizontal.*

6.5 Velocity of Conical Pendulum

Consider a conical pendulum consisting of a bob of mass m revolving without friction in a circle of radius r at a constant speed v on a string of length L at an angle of θ from the vertical. See Fig. 4. The angle θ is related to the velocity v as

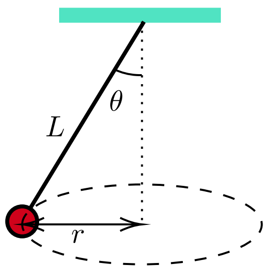

Fig. 4. Conical Pendulum

$$v = \sqrt{rg\tan\theta}$$

where g is acceleration due to gravity. If θ is increased, r is increased and consequently, the velocity is increased. When $\theta = \frac{\pi}{2}$, $r = L$ and hence

$$v = \sqrt{Lg\tan\frac{\pi}{2}} = \sqrt{Lg\frac{\sin\frac{\pi}{2}}{\cos\frac{\pi}{2}}} = \sqrt{Lg\frac{\cos\mathbf{0}}{\sin\mathbf{0}}} = \sqrt{\frac{Lg}{\mathbf{0}\left(1 - \frac{\mathbf{0}^2}{3!} + \frac{\mathbf{0}^4}{5!}\cdots\right)}}.$$

Since no further operation involving zero impends, we just write

$$v = \sqrt{\frac{Lg}{\mathbf{0}}}.$$

The velocity is endless or transfinite when $\theta = \frac{\pi}{2}$. The consequence of this is that *the string can never turn at the angle of* $\frac{\pi}{2}$ *to the vertical* **no matter how large the velocity of the object**.

7 Optics

7.1 Refraction of Light

That the ratio of two zeros can furnish a finite number different from unity finds an excellent illustration in a law of refraction of light. When light waves travelling in a transparent medium strike the surface of a second transparent medium, they tend to bend in order to follow the path of minimum time. This tendency is called refraction and is described by Snell's law of refraction

$$\frac{\sin\theta_1}{\sin\theta_2} = \frac{v_1}{v_2}$$

where θ_1 and θ_2 are the respective magnitudes of the incident angle in the first medium and the refracted angle in the second medium shown in Fig. 5, and v_1 and v_2 are velocities of light in the first and second media respectively.

If the light is made to pass along the normal in the first medium, it will not bend or refract in the second medium. It follows that the two angles are equal to zeros and consequently, the sines of their angles equal zeros. Since the ratio of the velocity of light in the first medium to the velocity of light in the second medium is a constant and finite, the ratio of zero from the sine of the incident angle to zero from the sine of the refracted angle equals a finite number.

I speculate an obvious objection which may be possibly advanced against my present undertaking. It may readily be imagined that the case of incident angle being zero is an exception to Snell's law since there is no refraction when light passes from medium 1 into medium 2 through the normal –thus it will probably be argued. Any doubt on my claim that zero incident angle is a case of Snell'a law may be set to rest by the definition of refractive index of the second medium with respect to the first medium, namely the ratio of the velocity of light in the first medium to the velocity of light in the second medium. Experimentally, this ratio is the same for any angle of incidence and this includes zero degree; for clearly the velocity of light changes as the light passes through the normal (represented by the dashed line) from medium 1 into medium 2.

7.2 Infinite Images in Parallel Mirrors

Let us place two plane mirrors such that they face each other and place an object between them. The number of images of the object we will see in the mirrors is undoubtedly endless: it is endless (transfinite) because we never finish counting them, that is, **no matter how far we have counted**

the images, *we shall never reach any end.* There is, therefore, no particular number of images formed by the two parallel plane mirrors.

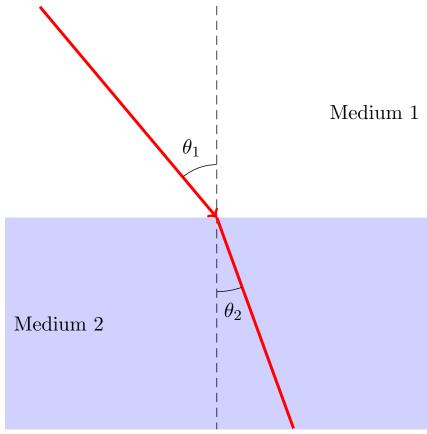

Fig. 5. Refraction of light

Let us use the Physics formula

$$n = \frac{360}{\theta} - 1$$

for the number n of images of any object placed between two plane mirrors inclined at an angle θ to each other. When the two mirrors face each other, $\theta = \mathbf{0}$ and n becomes zero divisor

$$\frac{360}{\mathbf{0}} - 1.$$

Since the number of images seen is endless (transfinite) and n is a zero divisor, it follows that the ratio of a finite number to zero is a transfinite number.

7.3 The Lens Equation

If f is the focal length of a convex lens and an object is placed at a distance x from the lens, then its image will be at a distance $y(x)$ from the lens, where the constant f, the independent variable x and the function $y(x)$ of x are related by the lens equation

$$\frac{1}{f} = \frac{1}{x} + \frac{1}{y(x)}$$

which, expressing $y(x)$ in terms of x and f, becomes

$$y(x) = \frac{fx}{x - f}.$$

This, applying our knowledge of partial fraction decomposition, becomes

$$y(x) = \frac{f^2}{x - f} + f.$$

If we set $x = f$, that is if the object is placed at the principal focus $F = F_2$ of the lens as shown in Fig. 6, we obtain the result

$$y(f) = \frac{f^2}{f - f} + f = \frac{f^2}{\mathbf{0}} + f.$$

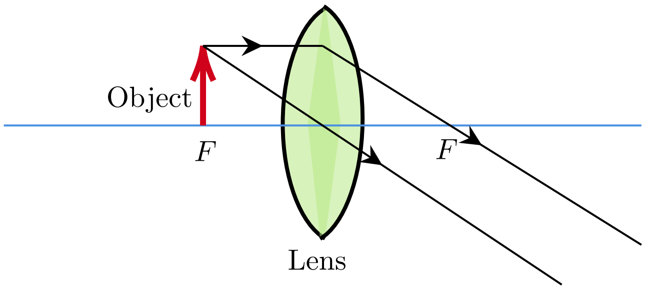

Fig. 6. Object at Focus $F = F_2$

This result shows that the image is formed at an unending distance $f^2/\mathbf{0} + f$ from the lens. But an endless distance has no end point. We therefore conclude that **no matter how far we trace the image** *we shall never reach the place where the image is formed.*

Now the magnification of the lens is

$$M(x) = \frac{y(x)}{x}$$

which, setting $x = f$, becomes

$$M(f) = \frac{\frac{f^2}{\mathbf{0}} + f}{f} = \frac{f}{\mathbf{0}} + 1.$$

The image is magnified an endless number of times. Suppose it were possible to the reach the image. Since endless sizes lack boundaries, we say that **no matter how large the measurement taken** *we shall never completely measure the size of the image.*

Take the middle O of the origin of the principal axis. This corresponds to the number **0**. Suppose an object is placed at P perpendicular to the principal axis of the lens such that it is nearer to the lens than its principal focus F. As the object is moved towards F, the image, virtual and erect, becomes large in size and its distance from the lens becomes great, i.e $y(x)$ is increased. When the object is just about to reach F, the image is just as well about to enlarge endlessly (infinitely) and its distance from the lens is just about to become endless (infinite) in extension. As long as the object has not reached F, the image's size and distance from the lens remains finite. But immediately the object reaches F exactly, the image's size and distance from the lens are at once thrown to the realm of the infinite, that is, they become endless in extensions.

It will be interesting to inquire into the nature of the image at infinity, though it is out of sight. If, in reality, $-\mathbf{0}$ coincides with $+\mathbf{0}$, two images, and not one, must be formed. Since infinity $1/\mathbf{0}$ comes in the pair $(+1/\mathbf{0}, -1/\mathbf{0})$, we have

$$y(f) = -\frac{f^2}{\mathbf{0}} + f$$

and

$$y(f) = +\frac{f^2}{\mathbf{0}} + f$$

From these we notice the presence of double images; one, virtual and erect, and positioned at $-f^2/\mathbf{0} + f$, the left side of the origin O; the other, real and inverted, and positioned at $+f^2/\mathbf{0} + f$, the right side of the origin O.

8 Conclusion

This paper made clear the notions of zero and infinity, providing examples from Physics to show how infinity may be interpreted whenever it arises.

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