Journal of Advances in Mathematics and Computer Science

35(8): 34-43, 2020; Article no.JAMCS.61310

ISSN: 2456-9968

(Past name: British Journal of Mathematics & Computer Science, Past ISSN: 2231-0851)

Using the Differential Transform Method to Solve Non-Linear Partial Differential Equations

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Authors' contributions

This work was carried out in collaboration between the two authors FAMM and FA.

Article Information

DOI: 10.9734/JAMCS/2020/v35i830311
Editor(s):
(1) Dr. Raducanu Razvan, Al. I. Cuza University, Romania.
Reviewers:
(1) Zain Fathi Abu-Sheer, Higher Institute of Engineering and Technology, Egypt.
(2) Mohamed El-Beltagy, Cairo University, Egypt.
(3) Chakravarthula S. K. Raju, Gandhi Institute of Technology and Management (GITAM), India.
Complete Peer review History: http://www.sdiarticle4.com/review-history/61310

Received: 22 July 2020
Accepted: 27 September 2020
Published: 20 November 2020

Original Research Article

Abstract

In this work, we reviewed the two-dimensional differential transform, and introduced the differential transform method (DTM). As an application, we used this technique to find approximate and exact solutions of selected non-linear partial differential equations, with constant or variable coefficients and compared our results with the exact solutions. This shows that the introduced method is very effective, simple to apply to linear and nonlinear problems and it reduces the size of computational work comparing with other methods.

Keywords: Two-dimensional differential transform; differential transform method; partial differential equations.

1 Introduction

The concept of differential transform was first proposed and applied to solve linear and nonlinear initial value problem in electric circuit analysis by Zhou [1]. The differential transform method is a semi-analytical technique used the Taylor series to construct the solutions of differential equations in the form of a power series. This technique is an iterative procedure for obtaining analytic series solutions of differential equations. It is also different from the high-order Taylor series method, which requires symbolic

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computation of the necessary derivatives of the data functions. The Taylor series method is computationally tedious for high order equations. In refs. [2,3,4,5,6], the two-dimensional differential transform method applied to quadratic Riccati's differential equation, and to solve a class of nonlinear optimal control problems, and to nonlinear second order pantograph equation and applied to nonlinear higher order boundary value problem. For more, applications see for examples [7-16]. The main aims in this work are to review the two-dimensional differential transform, and apply the two-dimensional differential transform method to selected nonlinear partial differential equations and compare our results with results obtained by other methods.

2 Two-Dimensional Differential Transform

The two-dimensional differential transform (DT) is defined as following:

2.1 Definition

Let $w(x,y)$ be an analytic function at (x_0, y_0), and if the Taylor series of $w(x,y)$ about (x_0, y_0) is given by:

$$w(x,y) = \sum_{h,k=0}^{\infty} \left[\frac{w^{(h,k)}(x)}{h_1!k!} \right]_{(x_0,y_0)} (x-x_0)^h (y-y_0)^k ,$$

then the two-dimensional differential transform (D_T) of the function $w(x,y)$ is defined by the formula

$$D_T\{w(x,y)\} = \left[\frac{w^{(k,h)}(x,y)}{k!h!} \right]_{(x_0,y_0)} := W(h,k) \tag{2.1}$$

In (2.1), $D_T\{w(x,y)\}$ represents the two-dimensional differential transform of $w(x,y)$ about (x_0, y_0) and it usually denoted by $W(k,h)$, and throughout our work, we take $(x_0, y_0) = (0,0)$. This reduces (2.1) to the formula

$$D_T\{w(x,y)\} = \left[\frac{w^{(k,h)}(x,y)}{k!h!} \right]_{(0,0)} . \tag{2.2}$$

2.2 Definition

The inverse differential transform is defined as

$$D_T^{-1}\{W(k,h)\} = w(x,y) := \sum_{k,h=0}^{\infty} W(k,h) x^k y^h \tag{2.3}$$

3 Useful Formulas and Properties

In this section, we list some properties and important formulas of the two-dimensional differential transform. The following properties can be prove by using (2.2) and (2.3):

Property (1): $D_T\left\{\alpha u(x,y)+\beta v(x,y)\right\}=\alpha D_T\left\{u(x,y)\right\}+\beta D_T\left\{v(x,y)\right\}$ (3.1)

Property (2): $D_T^{-1}\left\{\alpha U(k,h)+\beta V(k,h)\right\}=\alpha D_T^{-1}\left\{U(k,h)\right\}+\beta D_T^{-1}\left\{V(k,h)\right\}$ (3.2)

And the following formulas can be established by using, for example Refs. [7-14], and for more details see for example ref. [17].

Formula (1): $D_T\left\{w^{(r,s)}(x,y)\right\}=\dfrac{(k+r)!}{k!}\dfrac{(h+s)!}{h!}W(k+r,h+s)$ (3.3)

Formula (2): $D_T\left\{x^i y^j\right\}=\delta(k,i)\delta(h,j)=\begin{cases}1 & k=i \ \& \ h=j\\ 0 & otherwise\end{cases}$ (3.4)

Formula (3): If $w(x,y)=u(x,y)\cdot v(x,y)$, then

$$D_T\left\{w(x,y)\right\}=\sum_{r=0}^{k}\sum_{s=0}^{h}U(r,h-s)V(k-r,s) \tag{3.5}$$

Formula (4): If $w(x,y)=x^i y^j u(x,y)$, then

$$W(k,h)=\sum_{r=0}^{k}\sum_{s=0}^{h}\delta(r,i)\delta(h-s,j)U(k-r,s) \tag{3.6}$$

Formula (5): If $w(x,y)=u_x(x,y)\cdot u_x(x,y)$, then

$$W(k,h)=\sum_{r=0}^{k}\sum_{s=0}^{h}(r+1)(k-r+1)U(r+1,h-s)U(k-r+1,s) \tag{3.7}$$

Formula (6): If $w(x,y)=u(x,y)u_x(x,y)$, then

$$W(k,h)=\sum_{r=0}^{k}\sum_{s=0}^{h}(k-r+1)U(r,h-s)U(k-r+1,s) \tag{3.8}$$

Formula (7): If $w(x,y)=u(x,y)u_{xx}(x,y)$, then

$$W(k,h)=\sum_{r=0}^{k}\sum_{s=0}^{h}U(r,h-s)(k-r+2)(k-r+1)U(k-r+2,s) \tag{3.9}$$

Where $w(x,y)=\alpha u(x,y)+\beta v(x,y)$, $w^{(r,s)}(x,y)=\dfrac{\partial^{r+s}w(x,y)}{\partial x^r\partial y^s}$, and α, β are constants.

4 Numerical Examples

In this section, we apply differential transform method on selected non-linear partial differential equations, and compare our results with the exact solutions obtained by other methods.

4.1 Example

Consider the initial value problem

$$u_t = u^2 + u_x \tag{4.1}$$

$$u(x,0) = 1 + 2x . \tag{4.2}$$

To solve the initial value problem (4.1-2), [13], we first apply the two-dimensional transform on the both sides of (4.1), hence making use of (3.3) and (3.5), yield

$$(h+1)U(k,h+1) = \sum_{r=0}^{k}\sum_{s=0}^{h} U(r,h-s)\, U(k-r,s) + (k+1)U(k+1,h) \tag{4.3}$$

Now, for the initial condition (4.2), we have

$$\begin{aligned} &U(k,0) = \delta(k,0) + 2\delta(k,1), \qquad k = 0,1,2,\ldots \\ &U(0,0) = 1,\ U(1,0) = 2, \\ &U(k,0) = 0, \qquad k = 2,3,4,\ldots \end{aligned} \tag{4.4}$$

Next, (4.3), can be written in the form

$$U(k,h+1) = \frac{1}{(h+1)}\left[\sum_{r=0}^{k}\sum_{s=0}^{h} U(r,h-s)U(k-r,s) + (k+1)U(k+1,h)\right] \tag{4.5}$$

The iteration formula (4.5), we have

$$U(0,1) = 3, \qquad U(1,1) = 4, \qquad U(0,2) = 5, \quad U(2,1) = 4, \qquad U(1,2) = 14,\ldots$$

These leads to the series solution

$$u(x,t) = 1 + 2x + 3t + 4xt + 4x^2t + 5t^2 + 14xt^2 + \cdots \tag{4.6}$$

Hence, the given initial value problem satisfies the assumptions of the Cauchy-Kovalevsky theorem, and the Taylor series of exact solution can be found in ref. [15], page 107.

4.2 Example

Consider the initial value problem

$$u_t = uu_x \tag{4.7}$$

$$u(x,0)=1+x^2. \tag{4.8}$$

To solve the initial value problem (4.7-8), [12], we first apply the two-dimensional transform on the both sides of (4.7), hence making use of (3.3) and (3.8), yield

$$(h+1)U(k,h+1)=\sum_{r=0}^{k}\sum_{s=0}^{h}U(r,h-s)(k-r+1)U(k-r+1,s) \tag{4.9}$$

Now, for the initial condition (4.8), we have

$$\begin{aligned} &U(k,0)=\delta(k,0)+\delta(k,2) \qquad k=0,1,2,\dots \\ &U(0,0)=1, \qquad U(2,0)=1, \quad U(k,0)=0, \qquad k=1,3,4,\dots \end{aligned} \tag{4.10}$$

Next, we simplify equation (4.9) to

$$U(k,h+1)=\frac{1}{(h+1)}\left[\sum_{r=0}^{k}\sum_{s=0}^{h}U(r,h-s)(k-r+1)U(k-r+1,s)\right] \tag{4.11}$$

Now, the iteration formula (4.11), we have $U(1,1)=2,\ U(0,2)=1,\dots$

This leads to the series solution

$$u(x,t)=1+x^2+2xt+t^2+\cdots \tag{4.12}$$

Note that, the Taylor series (4.12) of exact solution can be found in ref. [15].

4.3 Example

Consider the initial value problem

$$\frac{\partial u}{\partial t}+\frac{1}{2}\frac{\partial u^2}{\partial x}-u(1-u)=0 \tag{4.13}$$

$$u(x,0)=e^{-x}, \tag{4.14}$$

The exact solution of initial value problem (4.13-14) is given by ref. [8] in the form

$$u(x,t)=e^{t-x}. \tag{4.15}$$

To solve the initial value problem (4.13-14), we first apply the two-dimensional transform on the both sides of (4.13), next making use of (2.2), (3.3), (3.5) and (3.8), yield

$$(h+1)U(k,h+1)+\frac{1}{2}\sum_{r=0}^{k}\sum_{s=0}^{h}U(r,h-s)(k-r+1)U(k-r+1,s)-$$

$$U(k,h)+\sum_{r=0}^{k}\sum_{s=0}^{h}U(r,h-s)\,U(k-r,s)=0 \tag{4.16}$$

For the initial condition (4.14), we have

$$U(k,0)=\frac{(-1)^k}{k!} \qquad k=0,1,2,\ldots \tag{4.17}$$

$$U(0,0)=1, \qquad U(1,0)=-1, \qquad U(2,0)=\frac{1}{2!},\ldots$$

Next, we simplify equation (4.16) to

$$U(k,h+1)=\frac{1}{(h+1)}\left[U(k,h)-\frac{1}{2}\sum_{r=0}^{k}\sum_{s=0}^{h}U(r,h-s)(k-r+1)U(k-r+1,s)-\sum_{r=0}^{k}\sum_{s=0}^{h}U(r,h-s)\,U(k-r,s)\right] \tag{4.18}$$

Now, the iteration formula (4.18), gives

$$U(0,1)=1, \qquad U(1,1)=-1, \qquad U(0,2)=\frac{1}{2!}, \quad U(2,1)=\frac{1}{2!}, \quad U(1,2)=\frac{-1}{2!},\ldots$$

This leads to the series solution

$$u(x,t)=1-x+t-xt+\frac{t^2}{2!}+\frac{x^2}{2!}+\frac{x^2t}{2!}-\frac{xt^2}{2!}+\cdots$$

$$=\left(1+t+\frac{t^2}{2!}+\cdots\right)\left(1-x+\frac{x^2}{2!}-\cdots\right). \tag{4.19}$$

Which is the Taylor series of the exact solution (4.15).

4.4 Example

Consider the initial value problem

$$u_t=u_xu_x+uu_{xx} \tag{4.20}$$

$$u(x,0)=\frac{1}{c}x^2. \tag{4.21}$$

To solve the initial value problem (4.20-21), [13], we first apply the two-dimensional transform on the both sides of (4.20), next making use of (3.3), (3.7) and (3.9), yield

$$\begin{aligned}(h+1)U(k,h+1) &= \sum_{r=0}^{k}\sum_{s=0}^{h}(r+1)(k-r+1)U(r+1,h-s)U(k-r+1,s) \\ &+\sum_{r=0}^{k}\sum_{s=0}^{h}U(r,h-s)(k-r+1)(k-r+2)U(k-r+2,s)\end{aligned} \tag{4.22}$$

Next, for the initial condition (4.21), we have

$$\begin{aligned}&U(k,0)=\frac{1}{c}\delta(k,2), \qquad U(2,0)=\frac{1}{c} \\ &U(k,0)=0, \qquad k=0,1,3,\ldots\end{aligned} \tag{4.23}$$

Next, we simplify equation (4.22) to

$$U(k,h+1)=\frac{1}{h+1}\left[\sum_{r=0}^{k}\sum_{s=0}^{h}(r+1)(k-r+1)U(r+1,h-s)U(k-r+1,s)+\sum_{r=0}^{k}\sum_{s=0}^{h}U(r,h-s)(k-r+1)(k-r+2)U(k-r+2,s)\right] \tag{4.24}$$

Now, the iteration formula (4.24), we have

$$U(2,1)=\frac{6}{c^2}, \qquad U(2,2)=\frac{36}{c^3}, \qquad U(2,3)=\frac{216}{c^4},\ldots$$

This leads to the series solution

$$u(x,t)=\frac{1}{c}x^2+\frac{6}{c^2}x^2t+\frac{36}{c^3}x^2t^2+\frac{216}{c^4}x^2t^3+\cdots$$

$$=x^2\left[\frac{1}{c}+\frac{6}{c^2}t+\frac{36}{c^3}t^2+\frac{216}{c^4}t^3+\cdots\right] \tag{4.25}$$

This result can be found in [18-pages-119-120] and [15].

4.5 Example

Consider the initial value problem

$$u_{tt}-uu_{xx}=1-\frac{x^2+t^2}{2} \qquad 0\le x,\ t\le 1 \tag{4.26}$$

$$u(x,0)=\frac{x^2}{2}, \tag{4.27}$$

$$u_t(x,0)=0. \tag{4.28}$$

The exact solution of this initial value problem is given in [14] as

$$u(x,t)=\frac{x^2+t^2}{2}. \tag{4.29}$$

To solve the initial value problem (4.26-27-28), we first apply the two-dimensional transform on the both sides of (4.26), hence making use of (3.3), (3.4) and (3.9), yield

$$(h+2)(h+1)U(k,h+2)-\sum_{r=0}^{k}\sum_{s=0}^{h}U(r,h-s)(k-r+2)(k-r+1)U(k-r+2,s)=$$
$$\delta(k,0)\delta(h,0)-\frac{1}{2}\delta(k,2)-\frac{1}{2}\delta(h,2). \tag{4.30}$$

Next, for the initial condition (4.27), we have

$$U(k,0)=\frac{1}{2}\delta(k,2) \qquad k=0,1,2,\ldots$$
$$U(2,0)=\frac{1}{2}, \qquad U(k,0)=0, \qquad k=0,1,3,\ldots \tag{4.31}$$

and, for the initial condition (4.28), we have

$$(h+1)U(k,h+1)=0$$
$$U(k,h+1)=0, \qquad k=0,1,2,\ldots$$
$$U(0,1)=0, \qquad U(1,1)=0, \qquad U(k,1)=0, \qquad k=2,3,\ldots \tag{4.32}$$

Also, we can simplify equation (4.30) to

$$U(k,h+2)=\frac{1}{(h+2)(h+1)}\left[\sum_{r=0}^{k}\sum_{s=0}^{h}U(r,h-s)(k-r+2)(k-r+1)\,U(k-r+2,s)+\right.$$
$$\left.\delta(k,0)\,\delta(h,0)-\frac{1}{2}\delta(k,2)-\frac{1}{2}\delta(h,2)\right] \tag{4.33}$$

Now, the iteration formula (4.33), gives $U(0,2)=\frac{1}{2}.$ and the others are zero. This leads to the solution.

$$u(x,t)=\frac{x^2+t^2}{2}. \tag{4.34}$$

5 Conclusion

In this work, we reviewed the two-dimensional differential transform and introduced the differential transform method (DTM). As an application, the two-dimensional differential transform method has been applied to selected non-linear differential equations. This study shows that the DTM reduces the computational difficulties comparing with the usual power series. These results reveal that this method is very efficient, simple and can be applied to many complicated non-linear partial differential equations.

Competing Interests

Authors have declared that no competing interests exist.

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http://www.sdiarticle4.com/review-history/61310