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Extension Rules of Newman–Janis Algorithm for Rotation Metrics in General Relativity

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The sole author designed, analysed, interpreted and prepared the manuscript.

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ABSTRACT

Aims: The aim of this study is to extend the formula of Newman–Janis algorithm (NJA) and introduce the rules of the complexifying seed metric. The extension of NJA can help determine more generalized axisymmetric solutions in general relativity.
Methodology: We perform the extended NJA in two parts: the tensor structure and the seed metric function. Regarding the tensor structure, there are two prescriptions, the Newman–Penrose null tetrad and the Giampieri prescription. Both are mathematically equivalent; however, the latter is more concise. Regarding the seed metric function, we propose the extended rules of a complex transformation by r^2/Σ and combine the mass, charge, and cosmologic constant into a polynomial function of r.

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Results: We obtain a family of axisymmetric exact solutions to Einstein's field equations, including the Kerr metric, Kerr–Newman metric, rotating–de Sitter, rotating Hayward metric, Kerr–de Sitter metric and Kerr–Newman–de Sitter metric. All the above solutions are embedded in ellipsoid-symmetric spacetime, and the energy-momentum tensors of all the above metrics satisfy the energy conservation equations.
Conclusion: The extension rules of the NJA in this research avoid ambiguity during complexifying the transformation and successfully generate a family of axisymmetric exact solutions to Einsteins field equations in general relativity, which deserves further study.

Keywords: Cosmological constant; ellipsoid coordinate transformation; Newman–Janis algorithm; Kerr metric; Kerr–de Sitter metric; Kerr–Newman metric; Kerr–Newman–de Sitter metric, rotating Hayward metric.

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1 INTRODUCTION

Einstein's field equations are a set of second-order hyperbolic partial differential equations with space and time as their independent variables and metric as their dependent variable. The equations are highly nonlinear that finding an exact solution is difficult, except by using symmetry. To discover new solutions to Einstein's field equations, reference [1] has proposed the Newman–Penrose (NP) formalism and Newman–Janis algorithm (NJA), whose direct result is the Kerr–Newman metric. NJA is a method known for solving an axisymmetric solution from a static one, which usually requires the NP formalism using complex null tetrads with ideas taken from two-component spinors.

In recent years, several studies have focused on the NJA, which is considered as the generation technique of field equations. It is a powerful tool that can generate the axisymmetric solution of a rotating black hole from the corresponding seed metric possessing spherical symmetry. In reference [2], it is proved that the black hole solution that can be generalized by the NJA is Petrov type D classification. However, in reference [3], it is indicated that NJA can also produce exact solutions as Petrov II fluid.

Generally, the NJA needs to perform a complex transformation with the NP formalism and null tetrad. The NP spin coefficients, Ricci-NP scalars and Weyl scalars can be given, which are useful for solving unknown analytical solutions. According to [4], the NJA can also be performed with Giampieri prescription, which is mathematically equivalent to but more concise than the NP null tetrad.

The NJA was later developed by [5] and used to solve more generalized non-stationary rotational black holes, such as the Vaidya–Kerr–Newman de Sitter metrics, and the variable cosmological constant function $\Lambda(u)$ to determine a non-stationary de–Sitter cosmological model. In recent years, [6] and [7] further extended the NJA to the rotation solution of an NUT charged black hole. However, Erbin's NJA generalization rules are the same as Newman's $\frac{1}{r}$ term but are ambiguous on r^2 or higher-order terms in the seed metric function.

The aim of this research is to extend the formula of the NJA and introduce consistent rules for the complexifying seed metric. The extension rules of the NJA can help in determine more generalized axisymmetric solutions in general relativity.

The rest of this paper is organized as follows: Section 2 presents the general procedures of NJA. Section 3 presents the extended formula of the NJA based on the seed metric function and analyses this technique to various axisymmetric solutions in general relativity. Section 4 presents the Kerr metric, Kerr–Newman metric, rotating de Sitter, rotating Hayward metric, Kerr–de Sitter metric and Kerr–Newman–de Sitter metric via this extended formula of NJA. Section 5, presents the tensor fields. Section 6 presents the discussions. In addition, this paper sets $c = G = 1$.

2 NEWMAN–JANIS ALGORITHM

The general procedure for the Newman-Janis algorithm can be summarized as follows:

1. Seclet the seed metric coordinates: Choose Eddington–Finkelstein coordinate representation to define the seed metric function $f(r)$.

2. Perform complex coordinate transformations and corresponding transformations:

(1) Coordinate tensor structure transformation: Tensor structure:that is, dx_μ (two prescriptions: Newman–Janis and Giampieri);

(2) Seed metric function transformation: The function $f(r)$ in the real domain is replaced by the complex function $\widetilde{f}(r,\overline{r})$.

3. The metric coordinates was changed according to the study's purpose, (e.g. Boyer–Lindquist coordinates or ellipsoid orthogonal coordinates).

3 EXTENSION FORMULATION OF THE NEWMAN–JANIS ALGORITHM

The static spherical symmetry metric of Einstein's theory of general relativity can be expressed in the following formulas:

$$\begin{aligned} ds^2 =& f dt^2 - f^{-1} dr^2 - r^2 d\Omega^2, \\ d\Omega^2 =& d\theta^2 + \sin^2\theta d\phi^2, \end{aligned} \tag{3.1}$$

where f is the seed metric defined as $f(r)$.

Step 1: Change the coordinates from (t, r, θ, ϕ) to (u, r, θ, ϕ).

Under the transformation $dt = du - f^{-1}dr$, Eq.(1) becomes the 'outgoing' Eddington–Finkelstein coordinates, which are given by

$$ds^2 = f du^2 + 2dudr - r^2 d\Omega^2 \tag{3.2}$$

Step 2: Complex transformation by NP formalism

In this step, the tensor structure dx_μ will undergo a complex coordinate transformation either by the Newman–Janis prescription or Giampieri prescription. Both the final results are the same, however, the latter is more concise.

2-1. Newman–Janis prescription: Newman–Penrose null complex tetrad

By introducing the formalism of null tetrad, the contravariant metric components can be written as

$$g^{\mu\nu} = l^\mu n^\nu + l^\nu n^\mu - m^\mu \overline{m}^\nu - m^\nu \overline{m}^\mu, \tag{3.3}$$

where the null tetrad vectors $(l^a, n^a, m^a, \overline{m}^a)$ and the gauge field A^a are

$$\begin{aligned} l^a =& \delta^a_r, \\ n^a =& \delta^a_u - \frac{f}{2}\delta^a_r, \\ m^a =& \frac{1}{\sqrt{2}r}\left(\delta^a_\theta + \frac{i}{\sin\theta}\delta^a_\phi\right), \\ \overline{m}^a =& \frac{1}{\sqrt{2}r}\left(\delta^a_\theta - \frac{i}{\sin\theta}\delta^a_\phi\right), \\ A^a =& -f_A l^a. \end{aligned} \tag{3.4}$$

The matrix form of the metric tensor $g_{\mu\nu}$ and its inverse matrix $g^{\mu\nu}$ are given by

$$\begin{aligned} g_{\mu\nu} =& \begin{pmatrix} f & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2\sin^2\theta \end{pmatrix}, \\ g^{\mu\nu} =& \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & -f & 0 & 0 \\ 0 & 0 & -r^{-2} & 0 \\ 0 & 0 & 0 & -r^{-2}\sin^{-2}\theta \end{pmatrix}. \end{aligned} \tag{3.5}$$

The following complex transformations are performed: $r, u \in R \rightarrow r', u' \in C$.

$$r \rightarrow r' - ia\cos\theta', u \rightarrow u' + ia\cos\theta', \theta \rightarrow \theta', \phi \rightarrow \phi'. \tag{3.6}$$

The transformed tetrads are listed as follows:

$$\begin{aligned} l'^a =&\delta^a_{r'}, \\ n'^a =&\delta^a_{u'} - \frac{\widetilde{f}}{2}\delta^a_{r'}, \\ m'^a =&\frac{1}{\sqrt{2}\left(r' + ia\cos\theta'\right)}\left(\delta^a_{\theta'} + \frac{i}{\sin\theta'}\delta^a_{\phi'} - \left(\delta^a_{u'} - \delta^a_{r'}\right) ia\sin\theta'\right), \\ \overline{m'}^a =&\frac{1}{\sqrt{2}(r' - ia\cos\theta')}\left(\delta^a_{\theta'} - \frac{i}{\sin\theta'}\delta^a_{\phi'} + \left(\delta^a_{u'} - \delta^a_{r'}\right) ia\sin\theta'\right), \\ A'^a =& - \widetilde{f}_A\left(du' - a\sin^2\theta d\phi\right). \end{aligned} \tag{3.7}$$

The seed metric function $f(r) \in R$ in the real domain is replaced with the complex function $\widetilde{f}(r, \overline{r})$:

$$r \to r' - ia\cos\theta', \overline{r} \to r' + ia\cos\theta', |r|^2 = r\overline{r} = \Sigma. \tag{3.8}$$

We obtain the complex transformed 'seed metric' function: $\widetilde{f}(r, \overline{r})$ can be written as the following rule. Removing the prime notation, we have

$$\widetilde{f}(r, \overline{r}) = 1 - \frac{2m(r)}{r}\left(\frac{r^2}{\Sigma}\right), \tag{3.9}$$

where $m(r)$ denotes the mass function as a function of r. The mass function can be a linear superposition of the mass M, charge Q, and cosmological constants,such that

$$m(r) = M - \frac{Q^2}{2r} + \frac{\Lambda r^3}{6} \tag{3.10}$$

The dimensionless ratio $\frac{r^2}{\Sigma}$ in Eq.(3.9) is the key to the NJA. It comes from the complex transformation rule $\frac{r}{2}\left(\frac{1}{r} + \frac{1}{\overline{r}}\right)$; when the rotational parameter a vanishes, the ratio is 1. By extending the definition of mass M to the mass function $m(r)$ in Eq.(3.10), this ratio can be applied to each term of the mass function and eliminate ambiguity in the previous literature.

After the complex transformation, $g^{\mu\nu}$ turns into $\widetilde{g}^{\mu\nu}$, and its inverse matrix $\widetilde{g}_{\mu\nu}$ are given by

$$\begin{aligned} \widetilde{g}^{\mu\nu} &= \begin{pmatrix} -\frac{a^2\sin^2\theta}{\Sigma} & 1 + \frac{a^2\sin^2\theta}{\Sigma} & 0 & -\frac{a}{\Sigma} \\ 1 & -\widetilde{f} & 0 & \frac{a}{\Sigma} \\ 1 + \frac{a^2\sin^2\theta}{\Sigma} & 0 & -\frac{1}{\Sigma} & 0 \\ -\frac{a}{\Sigma} & \frac{a}{\Sigma} & 0 & -\frac{1}{\Sigma\sin^2\theta} \end{pmatrix}, \\ \widetilde{g}_{\mu\nu} &= \begin{pmatrix} \widetilde{f} & 1 & 0 & a\sin^2\theta(1 - \widetilde{f}) \\ 1 & 0 & 0 & -a\sin^2\theta \\ 0 & 0 & -\Sigma & 0 \\ a\sin^2\theta(1 - \widetilde{f}) & -a\sin^2\theta & 0 & -\sin^2\theta\left[\left(r'^2 + a^2\right) + a^2\sin^2\theta\left(1 - \widetilde{f}\right)\right] \end{pmatrix}. \end{aligned} \tag{3.11}$$

where the line elements of the metric read

$$\begin{aligned} ds^2 =&\widetilde{f}du'^2 + 2du'dr' + 2a\sin^2\theta(1 - \widetilde{f})du'd\phi' - 2a\sin^2\theta dr'd\phi' \\ &- \Sigma d\theta^2 - \sin^2\theta\left[\left(r'^2 + a^2\right) + a^2\sin^2\theta\left(1 - \widetilde{f}\right)\right]d\phi'^2. \end{aligned} \tag{3.12}$$

2-2. Giampieri prescription

Next, we will introduce another transformation method wherein, the complex transformed metric can be obtained directly from the Eddington–Finkelstein coordinates in Eq.(3.2). The infinitesimal transformation is

$$\begin{aligned} dr \to& dr' + a\sin^2\theta d\phi', \\ du \to& du' - a\sin^2\theta d\phi'. \end{aligned} \tag{3.13}$$

Comparing it with the NP prescription, we have

$$\begin{aligned} dr &\to dr' + ia\sin\theta d\theta', \\ du &\to du' - ia\sin\theta d\theta'. \end{aligned} \tag{3.14}$$

Therefore, $id\theta = \sin\theta d\phi$ is the crucial rule of this prescription, and it can be proved that the results are exactly the same. After the above complex transformations, Eq. (3.2) is transformed as follows

$$\begin{aligned} ds^2 =& \widetilde{f}(du' - a\sin^2\theta d\phi')^2 + 2(du' - a\sin^2\theta d\phi') \\ & (dr' + a\sin^2\theta d\phi') - \Sigma(d\theta'^2 + \sin^2\theta d\phi'^2) \end{aligned} \tag{3.15}$$

Through simple algebraic calculation, it can be proven that metrics (3.15) and (3.12) are identical. The lengthy calculations of the null tetrad transformation are not performed in this prescription, ,and hence, it is more concise.

Step 3: Change coordinates

Example 1: Boyer–Lindquist coordinate system $(t_{BL}, r_{BL}, \theta_{BL}, \phi_{BL})$ According to [6], we have

$$\begin{aligned} du' &= dt_{BL} + g(r)dr' \\ d\phi' &= d\phi_{BL} + h(r)dr' \end{aligned} \tag{3.16}$$

Substituting into Eq.(3.12), let $g_{tr} = g_{r\phi} = 0$; then, we obtain

$$g(r) = -\frac{r^2 + a^2}{\Delta},\ h(r) = -\frac{a}{\Delta}, \tag{3.17}$$

where $\Delta = \widetilde{f}\Sigma + a^2\sin^2\theta$. Then, the following coordinate transformations are performed:

$$\begin{aligned} du' &= dt_{BL} - \frac{r^2 + a^2}{\Delta}dr', \\ d\phi' &= d\phi_{BL} - \frac{a}{\Delta}dr', \\ dr' &= dr_{BL}, \\ d\theta' &= d\theta_{BL}. \end{aligned} \tag{3.18}$$

Metric (3.12) is further rewritten using the Boyer–Lindquist coordinates, and the line elements and gauge field are listed as follows:

$$\begin{aligned} ds^2 =& \widetilde{f}dt_{BL}^2 - \frac{\Sigma}{\Delta}dr_{BL}^2 - \Sigma d\theta_{BL}^2 - X\sin^2\theta d\phi_{BL}^2 \\ & - 2a(\widetilde{f} - 1)\sin^2\theta dt_{BL}d\phi_{BL}, \\ X =& r'^2 + a^2 + ag_{t\phi} = r'^2 + a^2 + (1 - \widetilde{f})a^2\sin^2\theta, \\ \frac{\Sigma}{\Delta} =& g(r) - a\sin^2\theta h(r), \\ \Delta =& \widetilde{f}\Sigma + a^2\sin^2\theta, \\ A =& \widetilde{f}_A\left(dt_{BL} - a\sin^2\theta d\phi_{BL}\right). \end{aligned} \tag{3.19}$$

Example 2: Ellipsoid coordinate system (T, r, θ, Φ)

Our previous research [8, 9, 10] has shown that the off-diagonal term in the BoyerLindquist coordinate can be eliminated by an ellipsoid coordinate transformation and that the Kerr metric and KerrNewman metric can be derived using the ellipsoid coordinate transformation, which is given by

$$\begin{aligned} dT &= dt - a\sin^2\theta d\phi, \\ d\Phi &= d\phi - \frac{a}{r^2 + a^2}dt. \end{aligned} \tag{3.20}$$

According to [8], metric (3.19) can be rewritten in the ellipsoid coordinate system by using Eq.(3.20), then we have

$$\begin{aligned} ds^2 =& \frac{\Delta}{\Sigma}dT^2 - \frac{\Sigma}{\Delta}dr^2 - \Sigma d\theta^2 \\ & - \frac{(r^2 + a^2)^2\sin^2\theta}{\Sigma}d\Phi^2 \\ \Delta =& \widetilde{f}\Sigma + a^2\sin^2\theta \\ A =& \widetilde{f}_A dT \end{aligned} \tag{3.21}$$

From the above discussion, we complete the extension of NJA, where metric (3.12) is in the Eddington–Finkelstein coordinates, metric (3.19) is in the Boyer–Lindquist coordinates and metric (3.21) is in the ellipsoid orthogonal coordinates. We will introduce the applications of these metrics in the next section.

4 NEWMAN–JANIS ALGORITHM IN GENERAL RELATIVITY: APPLICATIONS

4.1 Flat Space

The NJA is applied to the Minkowski spacetime, and we let $f = 1$. The final result is obtained from the spherical polar coordinate (4.1) to the ellipsoid axisymmetric coordinates (4.2). [11] proposed the ellipsoid coordinate transformation method and its mathematical relation with the NJA.

$$ds^2 = dt^2 - dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2, \quad (4.1)$$

$$ds^2 = dt^2 - \frac{\Sigma}{r^2 + a^2} dr^2 - \Sigma d\theta^2 - (r^2 + a^2)\sin^2\theta d\phi^2. \quad (4.2)$$

This result is an important check of consistency, which will help us to extend the algorithm to more general cases.

4.2 Kerr Metric

The Schwarzschild metric is a vacuum static spherical symmetric solution in generalized relativity that can be used as a seed metric function.

$$f_S(r) = 1 - \frac{2m(r)}{r}, \;\; m(r) = M, \quad (4.3)$$

The complex transformation is performed using Eq.(3.9):

$$\begin{aligned} \widetilde{f}_K(r, \overline{r}) =& 1 - \frac{2M}{r}\left(\frac{r^2}{\Sigma}\right) = 1 - \frac{2Mr'}{\Sigma}, \\ X =& r'^2 + a^2 + \frac{2Mr'}{\Sigma} a^2 \sin^2\theta, \\ \Delta_K(r) =& \widetilde{f}\Sigma + a^2 \sin^2\theta = r'^2 + a^2 - 2Mr'. \end{aligned} \quad (4.4)$$

Substituting all the functions of Eq.(4.4) into the metric (3.19), the Kerr metric in BoyerLindquist form is obtained as follows

$$\begin{aligned} ds^2 =& (1 - \frac{2Mr}{\Sigma})dt^2 + \frac{4Mra\sin^2\theta}{\Sigma} dt d\phi - \frac{\Sigma}{\Delta} dr^2, \\ & - \Sigma d\theta^2 - (r^2 + a^2 + \frac{2Mra^2\sin^2\theta}{\Sigma})\sin^2\theta d\phi^2. \end{aligned} \quad (4.5)$$

By using Eq.(3.20), metric (4.5) can be rewritten into ellipsoid coordinate orthogonal form, which is given by.

$$ds^2 = \frac{\Delta_K}{\Sigma} dT^2 - \frac{\Sigma}{\Delta_K} dr^2 - \Sigma d\theta^2 - \frac{(r^2 + a^2)^2 \sin^2\theta}{\Sigma} d\Phi^2. \quad (4.6)$$

These derivations recover the Kerr metric. Clearly, when a vanishes, metrics (4.5) and (4.6) return to the Schwarzchild metric.

4.3 Kerr-Newman Metric

The seed metric function is the Reissner–Nordström metric, and the corresponding gauge field is

$$f_{RN}(r) = 1 - \frac{2m(r)}{r}, \; m(r) = M - \frac{Q^2}{2r}, \; f_A = \frac{Q}{r}. \quad (4.7)$$

The complex transformation is performed according to the rule of Eq.(3.9), and the transformed complex functions are given by

$$\begin{aligned}
\widetilde{f}_{KN}(r,\overline{r}) =&1-\frac{2Mr'-Q^2}{\Sigma},\\
\widetilde{f}_A =&\frac{Q}{r}\left(\frac{r^2}{\Sigma}\right)=\frac{Qr'}{\Sigma},\\
A =&\widetilde{f}_A(dt-a\sin^2\theta d\phi),\\
X =&r'^2+a^2+\frac{2Mr'-Q^2}{\Sigma}a^2\sin^2\theta,\\
\Delta_{KN}\left(r\right) =&\widetilde{f}\Sigma+a^2\sin^2\theta=r'^2+a^2-2Mr'+Q^2.
\end{aligned} \tag{4.8}$$

Substituting Eq.(4.8) into metric (3.21), the ellipsoid form of the Kerr–Newman metric reads

$$ds^2=\frac{\Delta_{KN}}{\Sigma}dT^2-\frac{\Sigma}{\Delta_{KN}}dr^2-\Sigma d\theta^2-\frac{(r^2+a^2)^2\sin^2\theta}{\Sigma}d\Phi^2. \tag{4.9}$$

When Q approaches zero, metrics (4.9) will return to the Kerr metric.

4.4 Rotating de Sitter Metric

The de Sitter metric is an exact solution to the vacuum Einstein's field equations with a cosmological constant, and the seed metric function is given by

$$f_\Lambda(r)=1-\frac{2m(r)}{r},\ m(r)=\frac{\Lambda r^3}{6}, \tag{4.10}$$

where Λ denotes the cosmological constant of the de Sitter space. This corresponds to the rotating de Sitter solution for $\Lambda>0$ and to the anti-de Sitter solution for $\Lambda<0$. Regarding the r^2 item, [6, 7] suggested the transformation rule $r^2\to\Sigma$. However, [12] shows that if the function $\frac{\Lambda r^2}{3}$ is converted to $\frac{\Lambda\Sigma}{3}$, the correct rotating de Sitter metric could not be obtained by the NJA. To avoid this error, [5] applied the Wang-Wu function. In this study we introduce the rules of $\frac{r^2}{\Sigma}$ to perform this complex transformation. The results are listed as follows

$$\begin{aligned}
\widetilde{f}_{R\Lambda}(r,\overline{r}) =&1-\frac{\Lambda r'^4}{3\Sigma},\\
X =&r'^2+a^2+\frac{\Lambda r'^4}{3\Sigma}a^2\sin^2\theta,\\
\Delta_{R\Lambda}\left(r\right) =&\widetilde{f}\Sigma+a^2\sin^2\theta=r'^2+a^2-\frac{\Lambda r'^4}{3}.
\end{aligned} \tag{4.11}$$

Substituting all the metric functions of Eq.(4.11) into Eq.(3.21), the Rotating de Sitter metric in ellipsoid coordinates is obtained by

$$ds^2=\frac{\Delta_{R\Lambda}}{\Sigma}dT^2-\frac{\Sigma}{\Delta_{R\Lambda}}dr^2-\Sigma d\theta^2-\frac{(r^2+a^2)^2\sin^2\theta}{\Sigma}d\Phi^2. \tag{4.12}$$

4.5 Rotating Hayward Metric

The Hayward metric is the simplest descriptor of non-singular black holes. This metric was proposed by [13]. It is a regular, static, spherically symmetric and asymptotically flat minimum model. This metric is not derived from any modified gravity theory, but it provides a framework to test the formation and evaporation of non-singular black holes in general relativity. The seed metric function is

$$f_H(r) = 1 - \frac{2m(r)}{r},\ m(r) = M\frac{r^3}{r^3+g^3}, \tag{4.13}$$

where g is a non-zero constant. Performing complex transformations using the rules of Eq. (3.9), we obtain

$$\begin{aligned} \widetilde{f}_{RH}(r,\overline{r}) =& 1 - \frac{2Mr'}{\Sigma}\left(\frac{r^3}{r^3+g^3}\right), \\ X =& r'^2 + a^2 + \frac{2Mr'}{\Sigma}\left(\frac{r^3}{r^3+g^3}\right)a^2\sin^2\theta, \\ \Delta_{RH}(r) =& \widetilde{f}\Sigma + a^2\sin^2\theta = r'^2 + a^2 - 2Mr'\left(\frac{r^3}{r^3+g^3}\right). \end{aligned} \tag{4.14}$$

Substituting all the metric functions in Eq. (4.14) into Eq.(3.21), we obtain the rotating Hayward metric in the ellipsoid form as

$$ds^2 = \frac{\Delta_{RH}}{\Sigma}dT^2 - \frac{\Sigma}{\Delta_{RH}}dr^2 - \Sigma d\theta^2 - \frac{(r^2+a^2)^2\sin^2\theta}{\Sigma}d\Phi^2. \tag{4.15}$$

4.6 Kerr–de Sitter Metric

In general relativity, the Schwarzschild–de Sitter metric describes a black hole in a de Sitter space with a cosmologic constant. It can be used as a seed metric function as

$$f_{S\Lambda}(r) = 1 - \frac{2m(r)}{r},\ m(r) = M + \frac{\Lambda r^3}{6}, \tag{4.16}$$

Using the rules of Eq.(3.9), we perform the complex transformation and obtain

$$\begin{aligned} \widetilde{f}_{K\Lambda}(r,\overline{r}) =& 1 - \frac{2Mr'}{\Sigma} - \frac{\Lambda r'^4}{3\Sigma}, \\ X =& r'^2 + a^2 + \frac{2Mr' + \frac{\Lambda r'^4}{3}}{\Sigma}a^2\sin^2\theta, \\ \Delta_{K\Lambda}(r) =& \widetilde{f}\Sigma + a^2\sin^2\theta = r'^2 + a^2 - 2Mr' - \frac{\Lambda r'^4}{3}. \end{aligned} \tag{4.17}$$

Substituting all the metric functions of Eq.(4.17) into Eq.(3.21), we obtain the Kerr–de Sitter metric in the ellipsoid form as

$$ds^2 = \frac{\Delta_{K\Lambda}}{\Sigma}dT^2 - \frac{\Sigma}{\Delta_{K\Lambda}}dr^2 - \Sigma d\theta^2 - \frac{(r^2+a^2)^2\sin^2\theta}{\Sigma}d\Phi^2. \tag{4.18}$$

4.7 Kerr–Newman–de Sitter Metric

The Reissner–Nordström–de Sitter metric is an exact solution to the Einstein–Maxwell equations in general relativity under a non-vanished cosmological constant Λ, which describes static mass and charge in a vacuum with spherical symmetry. It can be used as a seed metric function as follow:

$$f_{RN\Lambda}(r) = 1 - \frac{2m(r)}{r},\ m(r) = M - \frac{Q^2}{2r} + \frac{\Lambda r^3}{6},\ f_A = \frac{Q}{r}. \tag{4.19}$$

Performing complex transformations via rules of Eq.(3.9), we obtain

$$\begin{aligned}
\widetilde{f}_{KN\Lambda}(r,\overline{r}) =& 1-\frac{2Mr'}{\Sigma}+\frac{Q^2}{\Sigma}-\frac{\Lambda r'^4}{3\Sigma},\\
\widetilde{f}_A =& \frac{Q}{r}\left(\frac{r^2}{\Sigma}\right)=\frac{Qr'}{\Sigma},\\
A =& \widetilde{f}_A(dt-a\sin^2\theta d\phi),\\
X =& r'^2+a^2+\frac{2Mr'-Q^2+\frac{\Lambda r'^4}{3}}{\Sigma}a^2\sin^2\theta,\\
\Delta_{KN\Lambda}(r) =& \widetilde{f}\Sigma+a^2\sin^2\theta = r'^2+a^2-2Mr'+Q^2-\frac{\Lambda r'^4}{3}.
\end{aligned} \tag{4.20}$$

For the derivation of later sections, we substitute all the metric functions of Eq.(4.20) into Eq.(3.21). The Kerr–Newman–de Sitter metric tensor in the ellipsoid coordinates is given by

$$\begin{aligned}
ds^2 =& \frac{\Delta_{KN\Lambda}}{\Sigma}dT^2-\frac{\Sigma}{\Delta_{KN\Lambda}}dr^2-\Sigma d\theta^2-\frac{(r^2+a^2)^2\sin^2\theta}{\Sigma}d\Phi^2,\\
\Delta_{KN\Lambda}(r) =& r^2+a^2-2Mr+Q^2-\frac{\Lambda r^4}{3}.
\end{aligned} \tag{4.21}$$

The metric tensor under the BoyerLindquist coordinates is

$$g_{\mu\nu}=\begin{pmatrix}
\widetilde{f} & 0 & 0 & \frac{a\sin^2\theta\left(r'^2+a^2-\Delta\right)}{\Sigma}\\
0 & -\frac{\Sigma}{\Delta} & 0 & 0\\
0 & 0 & -\Sigma & 0\\
\frac{a\sin^2\theta(r'^2+a^2-\Delta)}{\Sigma} & 0 & 0 & -\sin^2\theta[(r'^2+a^2)+a^2\sin^2\theta(1-\widetilde{f})]
\end{pmatrix}. \tag{4.22}$$

5 TENSOR FIELD

5.1 Maxwell Tensor

Reference [14] showed that the extended NJA can be performed to obtain the NP scalar fields in rotating metrics. However, we will not further extend this method to de–Sitter spacetime owing to inconsistent transformation rules. To obtain the Maxwell tensor, we can begin with the definition. Using the BoyerLindquist coordinates, we have

$$\begin{aligned}
A_\mu =& \frac{Qr}{\Sigma}(dt-a\sin^2\theta d\phi),\\
F_{\mu\nu} =& \partial_\mu A_\nu-\partial_\nu A_\mu,\\
F^{\mu\nu} =& g^{\mu\rho}g^{\nu\sigma}F_{\mu\nu}.
\end{aligned} \tag{5.1}$$

Then, the Maxwell fields strength tensor associated with the Kerr–Newman–de Sitter solution can be determined as follows:

$$F_{\mu\nu}=\frac{Q}{\Sigma^2}\begin{pmatrix}
0 & 2r^2-\Sigma & -a^2r\sin 2\theta & 0\\
-(2r^2-\Sigma) & 0 & 0 & a\sin^2\theta(2r^2-\Sigma)\\
a^2r\sin 2\theta & 0 & 0 & -ar\sin 2\theta(r^2+a^2)\\
0 & -a\sin^2\theta(2r^2-\Sigma) & ar\sin 2\theta(r^2+a^2) & 0
\end{pmatrix}, \tag{5.2}$$

and the inverse matrix

$$F^{\mu\nu}=\frac{Q}{\Sigma^3}\begin{pmatrix}
0 & -(2r^2-\Sigma)(r^2+a^2) & a^2r\sin 2\theta & 0\\
(2r^2-\Sigma)(r^2+a^2) & 0 & 0 & a(2r^2-\Sigma)\\
-a^2r\sin 2\theta & 0 & 0 & -\frac{ar\sin 2\theta}{\sin^2\theta}\\
0 & -a(2r^2-\Sigma) & \frac{ar\sin 2\theta}{\sin^2\theta} & 0
\end{pmatrix}. \tag{5.3}$$

5.2 Energy-Momentum Tensor

Using the above Maxwell tensor and metric tensor, we can calculate the electromagnetic part of the energy-momentum tensor (EMT), which is given by

$$
\begin{aligned}
T_{\mu\nu} =& \frac{1}{4\pi}\left(F_{\mu\rho}F^{\rho}_{\nu} + \frac{1}{4}g_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}\right),\\
=& \frac{Q^2}{8\pi\Sigma}\begin{pmatrix} \frac{\Delta+a^2\sin^2\theta}{\Sigma^2} & 0 & 0 & \frac{-a\sin^2\theta}{\Sigma^2}A_{t\phi} \\ 0 & -\frac{1}{\Delta} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{-a\sin^2\theta}{\Sigma^2}A_{t\phi} & 0 & 0 & A_{\phi\phi} \end{pmatrix},\\
A_{t\phi} =& r^2 + a^2 + \Delta,\\
A_{\phi\phi} =& \frac{\sin^2\theta}{\Sigma^2}[(r^2+a^2)^2 + a^2\Delta\sin^2\theta],\\
\Delta =& r^2 + a^2 - 2Mr + Q^2 - \frac{r^4\Lambda}{3}.
\end{aligned}
\tag{5.4}
$$

Clearly, when Q vanishes, $T_{\mu\nu} = 0$, and if $\Lambda = 0, a = 0, M \neq 0, Q \neq 0$, then EMT (5.4) will return to the Reissner–Nordström's form. This step completes the complex transformation of the NJA, first given by Newman and Janis, and we extend the seed metric function, which gives more general rotation solutions to Einstein field equations. The extended NJA complex transformation rules in this study are summarised in Table 1. The details of our verification of Einstein's field equations with NP spin coefficients are listed in Appendix A.

6 DISCUSSION

The NJA is a powerful complex transformation method that can generate the axisymmetric solution from a 'seed metric'. In this study, the complex transformation rules and formulaic solutions of the $f(r)$ seed metric function are introduced. In addition, the study showed that it is more concise to derive rotation metrics by Giampieri prescription than by NP null tetrad, and the results are mathematically equal.

Second, the NJA procedure must treat the tensor structure and the seed metric functions separately. Regarding the tensor structure, the coordinate transformation is equivalent to the Minkovski metric from polar coordinates into ellipsoidal coordinates. Therefore, all the rotation metrics discussed above can be written in a consistent ellipsoidal symmetric orthogonal coordinate of Eq.(3.21). Regrading the $f(r)$ seed metric function, the rule of Eq.(3.9) is used to perform the complex transformation; thus the conversion rule is regarded as a crucial role in the extension of NJA. [15] proposed $\frac{Re\ r^{n+2}}{\Sigma}$, however, when $n = 0, f(r) = 1, \widetilde{f}(r,\bar{r}) = \frac{r^2}{\Sigma}$ this formula is not consistent. Therefore, in this study, we take $\frac{r^2}{\Sigma}$ times the mass function $m(r)$, and the results are fruitful. The rule of Eq.(3.9) comes from $\frac{r}{2}\left(\frac{1}{r}+\frac{1}{\bar{r}}\right)$, which is also equivalent to the ratio $\left(\frac{r}{r}\cdot\frac{r}{\bar{r}}\right)$.

By choosing the appropriate 'seed metric', the Kerr metric (with mass), Kerr–Newman metric (with mass and charge) and the rotating Hayward (without sigularity) can be derived. Furthermore, in the rotating universe with cosmological constants, the Kerr–de Sitter metric (with mass) and Kerr–Newman–de Sitter metric (with mass and charge) are derived by the extended NJA. The metric (4.21) of this research is similar with that reported by [5] and [14] but different from the results of [6], [16], [17], [18] and [19]. The results were derived from the same tensor structure transformation and may explain why Carters form of the Kerr–Newman–de Sitter metric (6.1) cannot be obtained by this extended NJA. If the cosmological constant $\Lambda = 0$, both metrics return to the Kerr–Newman metric, whereas if $a = 0$, both metrics return to the Reissner–Nordström–de Sitter metric.

$$\begin{aligned} ds^2 =& \frac{\Delta_r}{\Sigma}\left[\frac{dT}{\Xi}\right]^2 - \frac{\Sigma}{\Delta_r}dr^2 - \frac{\Sigma}{\Delta_\theta}d\theta^2 - \frac{(r^2+a^2)^2\sin^2\theta}{\Sigma}\Delta_\theta\left[\frac{d\Phi}{\Xi}\right]^2, \\ \Delta_r =& r^2+a^2-2Mr+Q^2-\frac{\Lambda r^2}{3}(r^2+a^2),\ \ \Delta_\theta = 1+\frac{\Lambda}{3}a^2\cos^2\theta, \\ \Xi =& 1+\frac{\Lambda a^2}{3}. \end{aligned} \tag{6.1}$$

Table 1. The extended NJA complex transformation rules

Tensor structure	$r \to r' - ia\cos\theta'$ $u \to u' + ia\cos\theta'$ $\theta \to \theta'$ $\phi \to \phi'$ $(l^\mu, n^\mu, m^\mu, \overline{m}^\mu) \to (l'^\mu, n'^\mu, m'^\mu, \overline{m'}^\mu)$
Seed metric $f(r) = 1 - \frac{2m(r)}{r}\left(\frac{r^2}{\Sigma}\right)$	$m(r) = M - \frac{Q^2}{2r} + \frac{\Lambda r^3}{6}$

7 CONCLUSION

The extension rules of the NJA in this research avoid ambiguity during complexifying the transformation and successfully generate a family of axisymmetric exact solutions to Einsteins field equations in general relativity. The tensor structure transformation of the NJA is proved to transform the spherical polar coordinate into the ellipsoid axisymmetric coordinate, therefore, all the exact solutions of the Kerr family can be further re-writing in the ellipsoid axisymmetric coordinate. However, whether this method can be applied to the interior Kerr solution in general relativity or modified gravity remains unclear and warrants further study.

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COMPETING INTERESTS

Author has declared that no competing interests exist.

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A APPENDIX: CONSERVATION EQUATIONS

In this appendix we verify the Einstein's filed equations $R_{ab} - (\frac{1}{2}R + \Lambda)g_{ab} = -KT_{ab}$ by the fact that the EMT (5.4) satisfies the conservation equations $T^{ab};_b = 0$. These include four equations, which can equivalently be written in three equations using Newman–Penrose spin coefficients. Then we have the following

$$D\rho^* = (\rho^* + p)(\rho + \bar{\rho}) + \omega\bar{\kappa}^* + \bar{\omega}\kappa^*, \tag{A.1}$$

$$\begin{aligned} D\mu^* + \bigtriangledown\rho^* + \bar{\delta}\omega + \delta\bar{\omega} =& \mu^* \left[(\rho + \bar{\rho}) - 2(\epsilon + \bar{\epsilon})\right] - (\rho^* + p)(\mu + \bar{\mu}) \\ & -\omega(2\pi + 2\bar{\beta} - \bar{\tau}) - \omega(2\bar{\pi} + 2\beta - \tau), \end{aligned} \tag{A.2}$$

$$D\omega + \delta p = \mu^*\kappa^* + (\rho^* + p)(\tau - \bar{\pi}) + \bar{\omega}\sigma - \omega(2\bar{\epsilon} - 2\rho - \bar{\rho}). \tag{A.3}$$

Where κ^*, τ, π, etc. are spin coefficients, and the derivative operators are defined as follows

$$D \equiv l^a\partial_a = \partial_r, \tag{A.4}$$

$$\bigtriangledown \equiv n^a\partial_a = \frac{1}{\Sigma}\left[(r^2 + a^2)\partial_u - \frac{\Delta_{KN\Lambda}}{2}\partial_r + a\partial_\phi\right], \tag{A.5}$$

$$\delta \equiv m^a\partial_a = \frac{1}{\sqrt{2}\bar{r}}\left[iasin\theta\partial_u + \partial_\theta + \frac{i}{sin\theta}\partial_\phi\right], \tag{A.6}$$

$$\bar{\delta} \equiv \bar{m}^a\partial_a = \frac{1}{\sqrt{2}r}\left[-iasin\theta\partial_u + \partial_\theta - \frac{i}{sin\theta}\partial_\phi\right]. \tag{A.7}$$

In order to verify whether components of EMT (5.4) with quantities μ^*, ρ^*, p and ω for the metric (4.21) satisfy these energy conservation equations (A.1)-(A.3) or not, the NP spin coefficients for the metric (4.21) are:

$$\kappa^* = \sigma = \lambda = \epsilon = \nu = 0, \tag{A.8}$$

$$\rho = -\frac{1}{r}, \ \ \mu = -\frac{1}{2r}\left[1 - \frac{2Mr + Q^2}{\Sigma} + \frac{a^2sin^2\theta}{\Sigma}\right], \tag{A.9}$$

$$\alpha = \frac{2ai - \bar{r}cos\theta}{2\sqrt{2}rrsin^2\theta}, \ \ \beta = \frac{cot\theta}{2\sqrt{2}\bar{r}}, \tag{A.10}$$

$$\pi = \frac{iasin\theta}{\sqrt{2}rr}, \ \ \tau = -\frac{iasin\theta}{\sqrt{2}\Sigma}, \tag{A.11}$$

$$\gamma = \frac{1}{2r\Sigma}\left[\left(r - M - \frac{2}{3}\Lambda r^3\right)r - \Delta_{KN\Lambda}\right], \tag{A.12}$$

$$\mu^* = \omega = \bar{\omega} = 0, \tag{A.13}$$

$$\rho^* = \frac{1}{K\Sigma^2}\left(Q^2 + \Lambda r^4\right), \tag{A.14}$$

$$p = \frac{1}{K\Sigma^2}\left[Q^2 - \Lambda r^2(r^2 + 2a^2cos^2\theta)\right]. \tag{A.15}$$

The equations (A.1)-(A.3) are verified by

$$D\rho^* = \frac{4\Lambda r^3a^2cos^2\theta - 4Q^2r}{K\Sigma^3}, \tag{A.16}$$

$$D\mu^* + \bigtriangledown\rho^* + \bar{\delta}\omega + \delta\bar{\omega} = \frac{2r\Delta_{KN\Lambda}(Q^2 - \Lambda r^2a^2cos^2\theta)}{K\Sigma^4} \tag{A.17}$$

$$D\omega + \delta p = \frac{4a^2cos\theta sin\theta\left(Q^2 - \Lambda r^2a^2cos^2\theta\right)}{\sqrt{2}Kr\Sigma^3}, \tag{A.18}$$

which can be shown equal to right hand side by using equations (A.4)-(A.15). It indicates that the EMT (5.4) satisfies the conversation equation $T^{ab}{}_{;b} = 0$. This shows that the Kerr–Newman–de Sitter metric (4.21) obtained by the extended NJA is an exact solution of Einstein's filed equations.

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