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Solution of Laplace Equation by Modified Differential Transform Method

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Authors' contributions

This work was carried out in collaboration among all authors. All authors read and approved the final manuscript.

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Abstract

In this paper, we applied the modified two-dimensional differential transform method to solve Laplace equation. Laplace equation is one of Elliptic partial differential equations. These kinds of differential equations have specific applications models of physics and engineering. We consider four models with two Dirichlet and two Neumann boundary conditions. The simplicity of this method compared to other iteration methods is shown here. It is worth mentioning that here only a few number of iterations are required to reach the closed form solutions as series expansions of some known functions.

Keywords: Differential transform method; elliptic partial differential equations; Laplace equation; iteration.

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1 Introduction

"In real world, many physical and natural phenomena are illustrated as differential equations. Most of the differential equations are nonlinear. So there difficulties in finding the exact or analytical solutions caused by the nonlinear part" [1,2]. There is a need for a method that handled nonlinear terms easily without any restrictions and less size of computations. Most of the problems in physics and engineering often use partial differential equations. A linear second order partial differential equation can be written as $Au_{xx} + Bu_{xy} + Cu_{yy} = F\left(x, y, u, u_x, u_y\right)$, where A, B and C may be functions of x and y. (x, y), denotes the independent variables and $u(x, y)$, the dependent variable, or solution of the PDE. If $B^2 - 4AC < 0$, then the equation is called Elliptic.

Laplace equation is one of Elliptic partial differential equations. The Laplace equation, named after the French mathematician Pierre-Simon Laplace. It is a fundamental equation in classical field theories and plays a crucial role in various scientific and engineering applications. The Laplace equation is a second-order partial differential equation. In two dimensions (2D): $\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$ Here, u is the unknown function of the variables x, y and ∇^2 is the Laplacian operator, which is the divergence of the gradient of u.

"Differential transform method [DTM] was first introduced in 1988 by Zhou. This method can be used to solve linear and non-linear ordinary differential equations. Chen and Ho developed this method as a two-dimensional differential transformation method for PDEs and obtains closure by rank solutions for linear and nonlinear initial value problems" [3]. "This method reduces the size of the computational domain and is easily applicable to many problems. The Dirichlet condition means the value of the function is prescribed, when $u(x, y) = g(x, y)$ on the boundary $\partial\Omega$" [4-6]. "The DTM is very effective numerical and analytical method for solving different types of differential equations as well as integral equations. This method converts the differential equations into recurrence relations, then by taylor series expansion with modified approach, we obtain convergent series solutions" [7-9]. Two dimensional DTM applied to solve initial value problems for pdes and compare the results with other methods and to get series solutions of partial differential equations [10].

Neumann condition means the value of the derivative normal to the boundary is prescribed, when $\frac{\partial u}{\partial n} = v(x, y)$ on the boundary $\partial\Omega$

Definition:

The differential transform method for one dimensional of a function y(x) is defined as:

$$Y(k) = \frac{1}{k!}\left[\frac{d^k y(x)}{dx^k}\right] x = 0, \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots. (a)$$

Where y(x) is the original function and Y(k) is the transformed function. Differential inverse transform of Y(k) is defined as

$$y(x) = \sum_{k=0}^{\infty} Y(k)x^k \approx y_N(x) = \sum_{k=0}^{N} Y(k)x^k \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots. (b)$$

By substituting equation(a) and (b) we get:

$$y(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!}\frac{d^k y(x)}{dx^k} | x = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots. (c)$$

Which implies that the concept of differential transforms are derived from taylor series expansion. In the previous definition we consider the case for x=0, but it is true for any fixed real number $x = x_0$

2 Methodology

Partial differential equations are used to formulate several phenomena in real world. There are many methods to solve pdes, one of these method is modified two dimensional DTM. In this study, the two dimensional differential transform method is used to find solutions of elliptic partial differential equations.

One dimensional differential transform function $u(x, y)$ can be represented as,

$$u(x, y) = \sum_{m=0}^{\infty} F(m)x^m \sum_{n=0}^{\infty} G(n)y^n = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} U(m, n)x^m y^n \quad (1)$$

Where $U(m, n) = F(m)G(m)$ is called spectrum of $u(x, y)$. $u(x, y) = f(x)g(y)$.

The basic definitions and operations of two-dimensional differential transform method:

<u>Definition 1:</u> If a function $u(x, y)$ is analytic and differentiable with respect to time t in the domain of interest,

$$U(m, n) = \frac{1}{m!n!}\left[\frac{\partial^{m+n}u(x,y)}{\partial x^m \partial y^n}\right]_{x=x_0, y=y_0} \quad (2)$$

Where the spectrum $U(m, n)$ is the transformed function. Then the differential inverse transform of $U(m, n)$ is defined as follows:

$$u(x, y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} U(m, n)(x - x_0)^m (y - y_0)^n \quad (3)$$

$u(x, y)$ - original function, $U(m, n)$ - transform function

Combining equation (2) and (3), it can be obtained that

$$u(x, y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{1}{m!n!}\left[\frac{\partial^{m+n}u(x,y)}{\partial x^m \partial y^n}\right]_{x=x_0, y=y_0} (x - x_0)^m (y - y_0)^n \quad (4)$$

<u>Theorem:</u>

1. If $u(x, y) = v(x, y) \pm w(x, y)$, then

$U(m, n) = V(m, n) \pm W(m, n)$

2. If $u(x, y) = av(x, y)$, then

$U(m, n) = aV(m, n)$

3. If $u(x, y) = v(x, y)w(x, y)$, then

$U(m, n) = \sum_{k=0}^{m} \sum_{l=0}^{n} V(k, n - l)W(m - k, l)$

4. If $u(x, y) = \frac{\partial^{r+s}v(x,y)}{\partial x^r \partial y^s}$, then

$U(m, n) = \frac{(m+r)!}{m!}\frac{(n+s)!}{n!}V(m + r, n + s)$

5. If $u(x, y) = e^{av(x,y)}$, then:

$$U(m, n) = \begin{cases} e^{av(0,0)}, & m = n = 0 \\ a\sum_{k=0}^{m-1}\sum_{l=0}^{n}\frac{m-k}{m} V(m - k, l)U(k, n - l) & m \geq 1 \\ a\sum_{k=0}^{m}\sum_{l=0}^{n-1}\frac{n-l}{n} V(k, n - l)U(m - k, n), & n \geq 1 \end{cases}$$

6. If $u(x,y) = x^k y^h$, then

$$U(m,n) = \begin{cases} \delta(m-k, n-h), & m = k, n = h \\ 0, & otherwise \end{cases}$$

7. If $u(x,y) = x^k e^{ay}$, then

$$U(m,n) = \delta(m-k)\frac{a^n}{n!}$$

3 Results

Solutions of Laplace equation:

Consider the second order Laplace equation, given as:

uxx + uyy = 0, $\qquad$ 0 < x, y < π

Dirichlet boundary condition (01)

u(x, 0) = sinh x, u(x, π) = −sinh x,

u(0, y) = 0, u(π, y) = sinh(π) cos y.

Taking the differential transform:

(m + 1)(m + 2)U(m + 2, n) + (n + 1)(n + 2)U(m, n + 2) = 0

u(x, 0) = $\sum_{m=0}^{\infty} U(m,0)x^m = sinhx = \sum_{m=0}^{\infty} \frac{x^m}{m!}$; m-odd

which, on comparing the both sides yield:

$$U(m,0) = \begin{cases} \frac{1}{m!}, & if\ m-odd \\ 0, & otherwise \end{cases}$$

$$u(0,y) = \sum_{n=0}^{\infty} U(0,n)y^n$$

$$U(0,n) = 0$$

$$U(m,n) = \begin{cases} \frac{(-1)^{n/2}}{m!\,n!}, & if\ m-odd, n-even \\ 0, & otherwise \end{cases}$$

$$u(x,y) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} U(m,n)x^m y^n$$

$$u(x,y) = \sum_{m=1,3,5..}^{\infty}\sum_{n=0,2,4,..}^{\infty} \frac{(-1)^{\frac{n}{2}}}{m!\,n!} x^m y^n$$

$$u(x,y) = sinhx\ cosy$$

Dirichlet boundary condition (02)

u(x, 0) = 0. u(x, π) = 0,

u(0, y) = sin y, u(π, y) = cosh(π) sin y.

Taking the differential transform

$$(m + 1)(m + 2)U(m + 2, n) + (n + 1)(n + 2)U(m, n + 2) = 0$$

$$u(x,0) = \sum_{m=0}^{\infty} x(m,0) = 0$$
$$u(m,0) = 0$$
$$u(o,y) = \sum_{n=0}^{\infty} x(m,0)x^m = siny = \sum_{n=0}^{\infty} \frac{(-1)^{\frac{n-1}{2}}}{n!} y^n$$
$$U(0,n) = \begin{cases} \frac{(-1)^{n-1/2}}{n!}, & if\ n-odd \\ 0, & otherwise \end{cases}$$

$$U(m,n) = \begin{cases} \frac{(-1)^{n-\frac{1}{2}}}{m!\,n!}, & if\ m-even, n-odd \\ 0, & otherwise \end{cases}$$
$$u(x,y) = \sum_{m=0.2,4.6,..}^{\infty} \sum_{n=1,3,5,..}^{\infty} \frac{(-1)^{n-1/2}}{n!} x^m y^n$$
$$u(x,y) = \sum_{m=0,2,4,..}^{\infty} \frac{x^m}{m!} \sum_{n=1,3,5}^{\infty} \frac{(-1)^{n-\frac{1}{2}}}{n!} y^n$$

$$u(x,y) = coshxsiny$$

<u>Neumann boundary condition (01)</u>

$u_y(x, 0) = 0$, $u_y(x, \pi) = 2 \cos 2x \sinh 2\pi$
$u_x(0, y) = 0$, $u_x(\pi, y) = 0$

Taking the differential transform

$$(m + 1)(m + 2)U(m + 2, n) + (n + 1)(n + 2)U(m, n + 2) = 0$$

$$u_y(x,\pi) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} n\pi^{n-1} U(m,n) x^m$$

$$u_y(x,\pi) = 2cos2xsinh(2\pi)$$

$$u_y(x,\pi) = 2 \sum_{m=0}^{\infty} \frac{(-1)^{\frac{m}{2}}}{m!} (2x)^m \sum_{n=0}^{\infty} \frac{(2\pi)^n}{n!}$$

By changing the index n , and comparison, we have

$$U(m,n+1) = \frac{(-1)^{m/2} 2^{m+n+1}}{(n+1)m!\,n!}$$

$$U(m,n) = \begin{cases} \frac{(-1)^{m/2} n^{m+n}}{m!\,n!}, & if\ m-even, n-odd \\ 0, & otherwise \end{cases}$$

$$u(x,y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(-1)^{\frac{m}{2}} 2^{m+n}}{m!\,n!} x^m y^n$$

$$u(x.y) = \sum_{m=0,2,4,\ldots}^{\infty} \frac{(-1)^{m/2}(2x)^m}{m!} \sum_{n=0,2,4,\ldots}^{\infty} \frac{(2y)^n}{n!}$$

$$u(x,y) = cos2xcosh2y$$

Neumann boundary condition (02)

uy(x, 0) = cos x, uy(x, π) = cosh π cos x,
ux(0, y) = 0, ux(π, y) = 0

Taking the differential transform

(m + 1)(m + 2)U(m + 2, n) + (n + 1)(n + 2)U(m, n + 2) = 0

$$u_y(x,0) = \sum_{n=0}^{\infty} U(m,1)x^m = cosx = \sum_{m=0}^{\infty} \frac{(-1)^{\frac{m}{2}}x^m}{m!}$$

$$U(m,1) = \begin{cases} \dfrac{(-1)^{m/2}}{m!}, & if\ m - even \\ 0, & otherwise \end{cases}$$

$$\mathrm{ux}(0,\mathrm{y}) = \sum_{n=0}^{\infty} U(1,n)y^n = 0$$

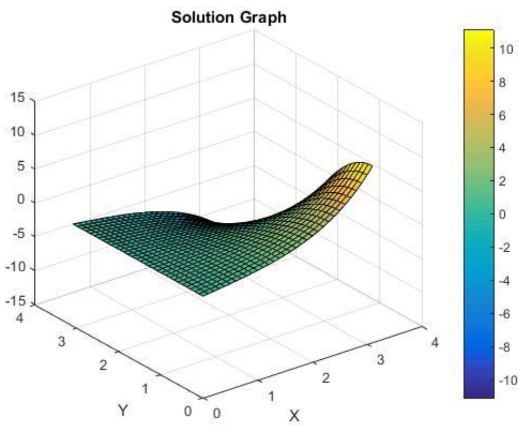
$$U(1,n) = 0$$

$$U(m,n) = \begin{cases} \dfrac{(-1)^{m/2}}{m!\,n!}, & if\ m - even, n - odd \\ 0, & otherwise \end{cases}$$

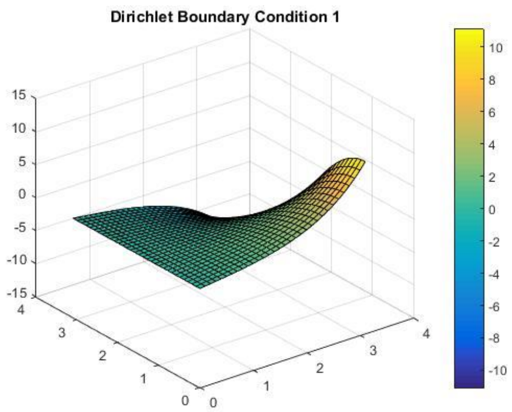
$$u(x,y) = \sum_{m=0,2,4,\ldots}^{\infty} \frac{(-1)^{\frac{m}{2}}x^m}{m!} \sum_{n=1,3,5,..}^{\infty} \frac{y^n}{n!}$$

$$u(x,y) = cosx.sinhy$$

Example 01(Dirichlet Boundary Condition 01):

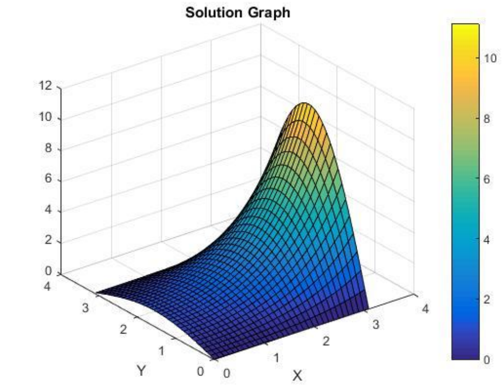


Approximate solution

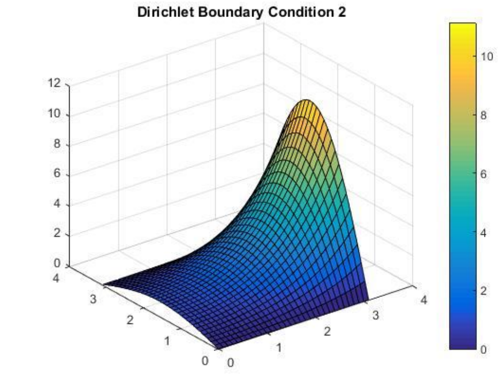


Exact Solution

<u>Example 02 (Dirichlet Boundary Condition 02):</u>

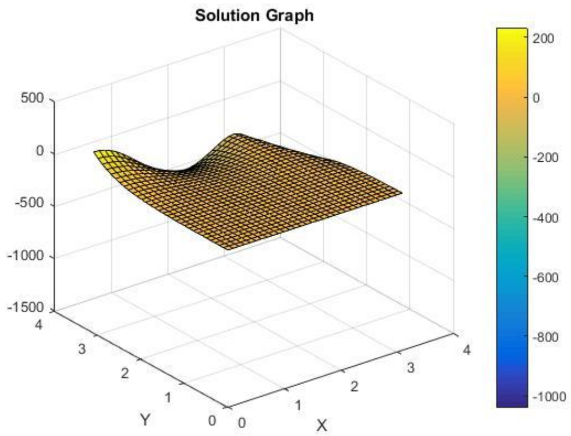


Approximate solution

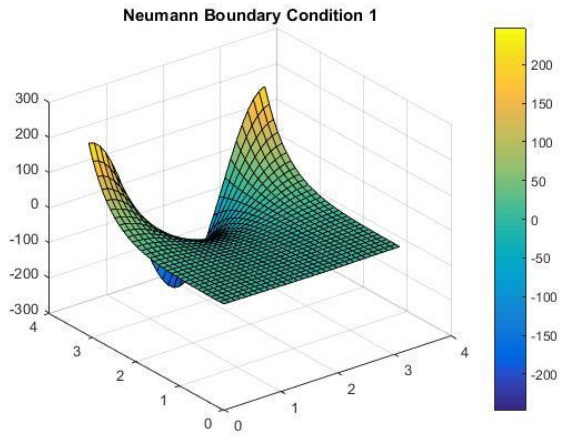


Exact Solution

<u>Example 03(Neumann Boundary Condition 01):</u>

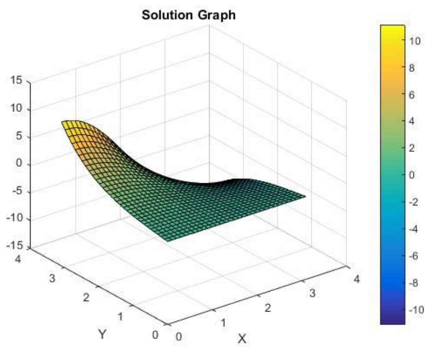


Approximate solution

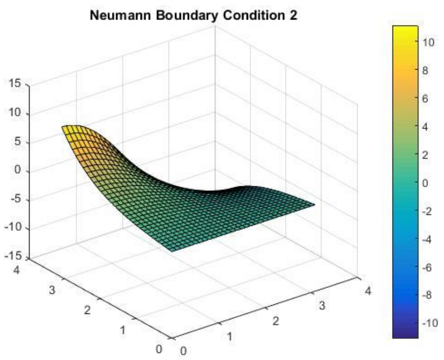


Exact Solution

<u>Example 04(Neumann Boundary Condition 02):</u>



Approximate solution



Exact Solution

4 Conclusion

In this paper, we have considered 4 cases of Laplace equation and presented solutions. We have successfully developed the DTM to obtain the exact solutions of Laplace equation. A computer program (MATLAB) is used in making the computation within some seconds provided that the program is well posed. In obtaining the Approximate solution, more accurate values can be obtained when using larger values of m and n. It is observed that it saves time and space. The results are consistent with the existing analytical solutions. It shows that it is effective and reasonable.

Competing Interests

Authors have declared that no competing interests exist.

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