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On hyper-singular integrals

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Abstract: The integrals $\int_{-\infty}^{\infty} t_+^{\lambda-1}\phi(t)dt$ and $\int_0^t (t-s)^{\lambda-1}b(s)ds$ are considered, $\lambda \neq 0, -1, -2...$, where $\phi \in C_0^\infty(\mathbb{R})$ and $0 \leq b(s) \in L^2_{loc}(\mathbb{R})$. These integrals are defined in this paper for $\lambda \leq 0$, $\lambda \neq 0, -1, -2, ...$, although they diverge classically for $\lambda \leq 0$. Integral equations and inequalities are considered with the kernel $(t-s)_+^{\lambda-1}$.

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1. Introduction

In [1] the following integral equation is of interest;

$$b(t) = b_0(t) + \int_0^t (t-s)^{\lambda-1}b(s)ds, \tag{1}$$

where b_0 is a smooth functions rapidly decaying with all its derivatives as $t \to \infty$, $b_0(t) = 0$ if $t < 0$. We are especially interested in the value $\lambda = -\frac{1}{4}$, because of its importance for the Navier-Stokes theory, [1], Chapter 5, [2,3]. The integral in (1) diverges in the classical sense for $\lambda \leq 0$. Our aim is to define this hyper-singular integral. There is a regularization method to define singular integrals $J := \int_{\mathbb{R}} t_+^\lambda \phi(t)dt$, $\lambda < -1$, in distribution theory, [4]. However, the integral in (1) is a convolution, which is defined in [4], p.135, as a direct product of two distributions. This definition is not suitable for our purposes because although $t_+^{\lambda-1}$ for $\lambda \leq 0$, $\lambda \neq 0, -1, -2, ...$ is a distribution on the space $C_0^\infty(\mathbb{R}_+)$ of the test functions, but it is not a distribution in the space $K = C_0^\infty(\mathbb{R})$ of the test functions used in [4]. Indeed, one can find $\phi \in K$ such that $\lim_{n\to\infty} \phi_n = \phi$ in K, but $\lim_{n\to\infty} \int_{\mathbb{R}} t_+^{\lambda-1}\phi(t)dt = \infty$ for $\lambda \leq 0$, so that $t_+^{\lambda-1}$ is not a linear bounded functional in K, i.e., not a distribution. On the other hand, one can check that $t_+^{\lambda-1}$ for $\lambda \in R$ is a distribution (a bounded linear functional) in the space $\mathcal{K} = C_0^\infty(\mathbb{R}_+)$ with the convergence $\phi_n \to \phi$ in $\mathcal{K}$ defined by the requirements: a) the supports of all ϕ_n belong to an interval $[a, b]$, $0 < a \leq b < \infty$, b) $\phi_n^{(j)} \to \phi^{(j)}$ in $C([a, b])$ for all $j = 0, 1, 2, ...$ Indeed, the functional $\int_0^\infty t_+^\lambda \phi(t)dt$ is linear and bounded in $\mathcal{K}$:

$$\left| \int_0^\infty t_+^\lambda \phi_n(t)dt \right| \leq (a^\lambda + b^\lambda) \int_a^b |\phi_n(t)|dt.$$

A similar estimate holds for the derivatives of ϕ_n. Although $t_+^{-\frac{5}{4}}$ is a distribution in $\mathcal{K}$, the convolution

$$h := \int_0^t (t-s)^{-\frac{5}{4}} b(s)ds := t_+^{-\frac{5}{4}} \star b \tag{2}$$

cannot be defined similarly to the definition in [4] because the function $\int_0^\infty \phi(u+s)b(s)ds$ does not, in general, belong to $\mathcal{K}$ even if $\phi \in \mathcal{K}$.

Let us define the convolution h using the Laplace transform

$$L(b) := \int_0^\infty e^{-pt}b(t)dt, \quad Rep > 0.$$

Laplace transform for distributions is studied in [5]. One has $L(t_+^{-\frac{5}{4}} \star b) = L(t_+^{-\frac{5}{4}})L(b)$. To define $L(t^{\lambda-1})$ for $\lambda \leq 0$, note that for $\mathrm{Re}\lambda > 0$ the classical definition

$$\int_0^\infty e^{-pt} t^{\lambda-1} dt = \frac{\Gamma(\lambda)}{p^\lambda} \tag{3}$$

holds. The right-side of (3) admits analytic continuation to the complex plane of λ, $\lambda \neq 0, -1, -2, ...$ This allows one to define integral (3) for any $\lambda \neq 0, -1, -2, ...$ Recall that the gamma function $\Gamma(\lambda)$ has its only singular points, the simple poles, at $\lambda = -n$, $n = 0, 1, 2, ...$ with the residue at $\lambda = -n$ equal to $\frac{(-1)^n}{n!}$. It is known that $\Gamma(z+1) = z\Gamma(z)$, so

$$\Gamma(-\frac{1}{4}) = -4\Gamma(3/4) := -c_1, \quad c_1 > 0. \tag{4}$$

Therefore, we define h by defining $L(h)$ as follows:

$$L(h) = -c_1 p^{\frac{1}{4}} L(b), \quad \lambda = -\frac{1}{4}, \tag{5}$$

and assume that $L(b)$ can be defined. That $L(b)$ is well defined in the Navier-Stokes theory follows from the a priori estimates proved in [1], Chapter 5. From (5) one gets

$$L(b) = -c_1^{-1} p^{-\frac{1}{4}} L(h). \tag{6}$$

2. Convolution of special functions

Define $\Phi_\lambda = \frac{t_+^{\lambda-1}}{\Gamma(\lambda)}$.

Lemma 1. *For any $\lambda, \mu \in \mathbb{R}$ the following formulas hold;*

$$\Phi_\lambda \star \Phi_\mu = \Phi_{\lambda+\mu}, \quad \Phi_{\lambda+0} \star \Phi_{-\lambda} = \delta(t). \tag{7}$$

Proof. For $\mathrm{Re}\lambda > 0$, $\mathrm{Re}\mu > 0$ one has

$$\Phi_\lambda \star \Phi_\mu = \frac{1}{\Gamma(\lambda)\Gamma(\mu)} \int_0^t (t-s)^{\lambda-1} s^{\mu-1} ds = \frac{t_+^{\lambda+\mu-1}}{\Gamma(\lambda)\Gamma(\mu)} \int_0^1 (1-u)^{\lambda-1} u^{\mu-1} du = \frac{t_+^{\lambda+\mu-1}}{\Gamma(\lambda+\mu)}, \tag{8}$$

where we used the known formula for beta function:

$$B(\lambda, \mu) := \int_0^1 u^{\lambda-1}(1-u)^{\mu-1} du = \frac{\Gamma(\lambda)\Gamma(\mu)}{\Gamma(\lambda+\mu)}.$$

Analytic properties of beta function follow from these of Gamma function. The function $\frac{1}{\Gamma(z)}$ is entire function of z.

Let us now prove the second formula (7). We have $\Gamma(\epsilon) \sim \epsilon$ as $\epsilon \to 0$. Therefore

$$\frac{t_+^{\lambda+\epsilon-\lambda-1}}{\Gamma(\epsilon)} \sim \epsilon t_+^{\epsilon-1}. \tag{9}$$

If f is any continuous rapidly decaying function then

$$\lim_{\epsilon\to 0} \epsilon \int_0^\infty t^{\epsilon-1} f(t) dt = f(0). \tag{10}$$

Indeed, fix a small $\delta > 0$, such that $f(t) \sim f(0)$ for $t \in [0, \delta]$ as $\delta \to 0$. Then, as $\epsilon \to 0$, one has

$$\lim_{\epsilon\to+0} \epsilon \int_0^\delta t^{\epsilon-1} f(t) dt = \lim_{\epsilon\to+0} \epsilon f(0) \frac{t^\epsilon}{\epsilon}\Big|_0^\delta = f(0) \lim_{\epsilon\to+0} \delta^\epsilon = f(0). \tag{11}$$

Note that

$$\lim_{\epsilon \to 0} \epsilon \int_{\delta}^{\infty} t^{\epsilon-1} f(t)dt = 0, \quad \delta > 0, \tag{12}$$

because $|\int_{\delta}^{\infty} t^{\epsilon-1} f(t)dt| \leq c$ and $\epsilon \to 0$. From (11) and (12) one obtains (10). So, the second formula (7) is proved. Lemma 1 is proved. □

Remark 1. The first formula (7) of Lemma 1 is proved in [4], pp.150–151. Our proof of the second formula (7) differs from the proof in [4] considerably.

Remark 2. A different proof of Lemma 1 can be given: $L(\Phi_\lambda \star \Phi_\mu) = \frac{1}{p^{\lambda+\mu}}$ by formula (3), and $L^{-1}(\frac{1}{p^{\lambda+\mu}}) = \Phi_{\lambda+\mu}(t)$. If $\lambda = -\mu$, then $\frac{1}{p^{\lambda+\mu}} = 1$ and $L^{-1}(1) = \delta(t)$.

3. Integral equation and inequality

Consider equation (1) and the following inequality:

$$q(t) \leq b_0(t) + t_+^{\lambda-1} \star q, \quad q \geq 0. \tag{13}$$

Theorem 1. *Equation (1) has a unique solution. This solution can be obtained by iterations by solving the Volterra equation*

$$b_{n+1} = -c_1^{-1}\Phi_{1/4} \star b_n + c_1^{-1}\Phi_{1/4} \star b_0, \quad b_{n=0} = c_1^{-1}\Phi_{1/4} \star b_0, \quad b = \lim_{n\to\infty} b_n. \tag{14}$$

Proof. Applying to (1) the operator $\Phi_{1/4}\star$ and using the second equation (7) one gets a Volterra equation

$$\Phi_{1/4} \star b = \Phi_{1/4} \star b_0 - c_1 b, \quad c_1 = |\Gamma(-\frac{1}{4})|,$$

or

$$b = -c_1^{-1}\Phi_{1/4} \star b + c_1^{-1}\Phi_{1/4} \star b_0, \quad c_1 = 4\Gamma(3/4). \tag{15}$$

The operator Φ_λ with $\lambda > 0$ is a Volterra-type equation which can be solved by iterations, see [1], p.53, Lemmas 5.10, 5.11. If $b_0 \geq 0$ then the solution to (1) is non-negative, $b \geq 0$. Theorem 1 is proved. □

For convenience of the reader let us prove the results mentioned above.

Lemma 2. *The operator $Af := \int_0^t (t-s)^p f(s)ds$ in the space $X := C(0,T)$ for any fixed $T \in [0,\infty)$ and $p > -1$ has spectral radius $r(A)$ equal to zero, $r(A) = 0$. The equation $f = Af + g$ is uniquely solvable in X. Its solution can be obtained by iterations*

$$f_{n+1} = Af_n + g, \quad f_0 = g; \quad \lim_{n\to\infty} f_n = f, \tag{16}$$

for any $g \in X$ and the convergence holds in X.

Proof. The spectral radius of a linear operator A is defined by the formula $r(A) = \lim_{n\to\infty} \|A^n\|^{1/n}$. By induction one proves that

$$|A^n f| \leq t^{n(p+1)} \frac{\Gamma^n(p+1)}{\Gamma(n(p+1)+1)} \|f\|_X, \quad n \geq 1. \tag{17}$$

From this formula and the known asymptotic of the gamma function the conclusion $r(A) = 0$ follows. The convergence result (16) is analogous to the well known statement for the assumption $\|A\| < 1$. Lemma 2 is proved. □

If $q \geq 0$ then inequality (13) implies

$$q \leq -c_1^{-1}\Phi_{1/4} \star q + c_1^{-1}\Phi_{1/4} \star b_0. \tag{18}$$

Inequality (18) can be solved by iterations with the initial term $c_1^{-1}\Phi_{1/4} \star b_0$. This yields

$$q \leq b, \tag{19}$$

where b solves (1). See also [6,7].

Conflicts of Interest: "The author declares no conflict of interest."

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