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Refining the Submean Inequality for Subharmonic Functions

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Abstract

It is known that the composition of a convex, increasing functional with a subharmonic function is subharmonic.
In this paper we show that the composition of a superquadratic functional with a subharmonic function is subharmonic, with a sharper submean inequality.
It is further demonstrated that the composition of an increasing convex functional with a non-negative superquadratic functional with a subharmonic function is subharmonic, with a sharper submean inequality.

Keywords: Convex function; subharmonic function; superquadratic function; jensen's inequality; submean inequality and refined jensen's inequality.

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1 Introduction

As an important tool, the Jensen's inequality has proved very useful in many areas of application such as in statistical physics, information theory, Rao-Blackwell theorem, etc. Many other important

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inequalities can also be obtained from the Jensen's inequality, such as the Hölder's, Minkwoski and Hadamard inequalities.

The discrete version of Jensen's inequality for functions as given in [1] is

$$\phi(\frac{1}{P_n}\sum_{i=1}^{n}p_ix_i) \leq \frac{1}{P_n}\sum_{i=1}^{n}p_i\phi(x_i), \tag{1.1}$$

where ϕ is a convex function defined on an interval I in $\mathbb{R}$, $(x_1,...,x_n) \in I^n(n \geq 2)$ and $(p_1,...,p_n)$ is any non-negative n-tuple satisfying $P_n = \sum_{i=1}^{n} p_i$.

A refinement of the Jensen's inequality is obtained upon replacing the convex function with a superquadratic function. This has also led to the refinement of many classical inequalities.

The continuous version of the refined Jensen's inequality as given in [2] is

$$\varphi(\int f d\mu) \leq \int[\varphi(f(s)) - \varphi(|f(s) - \textstyle\int f d\mu|)]d\mu(s), \tag{1.2}$$

for all probability measures μ and all non-negative, μ-integrable functions f.

Definition 1.1. [2] A function $\varphi : [0,\infty) \to \mathbb{R}$ is superquadratic provided that for all $x \geq 0$ there exists a constant $P_x \in \mathbb{R}$ such that

$$\varphi(y) \geq \varphi(x) + (y-x)P_x + \varphi(|y-x|) \tag{1.3}$$

for all $y \geq 0$.

We refer the reader to $[2, 3]$ for some general properties of superquadratic functions.

We state a well-known fact about the composition of a convex functional with a subharmonic function.

Theorem 1.2. [4] Let $-\infty \leq a < b \leq \infty$, and let $u : U \to [a,b)$ be a subharmonic function on an open set U in the complex plane . Let $\phi : (a,b) \to \mathbb{R}$ be an increasing convex function. Then $\phi \ o \ u$ is subharmonic on U,where

$$\phi(a) = \lim_{t\to a}\phi(t).$$

In this paper, we prove similar results on the composition of a superquadratic functional with a subharmonic function. The new inequalities obtained refine the usual submean inequality.

2 Preliminary Definitions and Results

In this section, we shall review some well known results about subharmonic and superquadratic functions.

2.1 Holomorphic Functions

Holomorphic functions are defined on an open subsets U of the complex plane , with values in and are complex-differentiable at every point in U. The complex- differentiability is a much stronger condition than the real- differentiability. Holomorphic functions are infinitely differentiable and can be expressed in a Taylor series. The term analytic function is often used interchangeably with holomorphic functions, further more, the class of analytic functions coincides with the class of holomorphic functions. Holomorphic functions are sometimes called regular functions. An entire function is a function which is holomorphic on the whole of .

Definition 2.1. [5] Let f be a complex-valued function defined on a domain D in ^{n}. If f satisfies the following two conditions:

(i) f is continuous in D and

(ii) f, has partial derivatives $\frac{\partial f}{\partial z_j}(j = 1, 2, ..., n)$ in D,

then f is a holomorphic function on D.

If continuity of the partial derivatives is not given, the converse is not necessarily true. Every holomorphic function can be separated into it's real and imaginary parts, and each of these is a solution of Laplace's equation on $\mathbb{R}^2$, thus given a holomorphic function $f(z) = u(x, y) + iv(x, y)$, both u and v are harmonic functions. Harmonic functions are both subharmonic and superharmonic. The subharmonic functions exhibit most of the properties of the harmonic functions as well as the properties of the holomorphic functions.

Before we look at the definition and some properties of subharmonic functions, we briefly discuss some functions that exhibit weaker notions of continuity.

2.2 Upper Semicontinuous Functions

There are functions that are not continuous, but exhibit a weaker property that ensures some of the properties of continuous functions.

Definition 2.2. [4] Let X be a topological space and $u : X \to [-\infty, \infty)$.

1. u is *upper semicontinuous* if for each $\alpha \in \mathbb{R}$, the set
   $\{x \in X : u(x) < \alpha\}$ is open.
2. u is *lower semicontinuous* if for each $\alpha \in \mathbb{R}$, the set
   $\{x \in X : u(x) > \alpha\}$ is open.

It is known that u is continuous if for all open intervals $(\alpha, \beta) \in \mathbb{R}, u^{-1}(\alpha, \beta)$ is open in X, thus $\{x \in X : \alpha < u(x) < \beta\}$ is open.

An equivalent formulation of the upper and lower semicontinuous functions follows as:

Theorem 2.3. [3, 6]

1. *u is upper semicontinuous iff* $\forall x \in X, \limsup_{y\to x} u(y) \leq u(x)$.
2. *u is lower semicontinuous iff* $\forall x \in X, \liminf_{y\to x} u(y) \geq u(x)$.
3. *u is lower semicontinuous iff $-u$ is upper semicontinuous.*
4. *u is continuous iff it is both upper and lower semicontinuous.*

An upper semicontinuous function is the limit of a decreasing sequence of continuous functions.

Theorem 2.4. [3, 7] Let u be an upper semicontinuous function on a metric space (X, d) and suppose that u is bounded above on X. Then there exists a sequence $(\phi_n)_{n\in\mathbb{N}}$ of continuous functions, $\phi_n : X \to \mathbb{R}$ such that, $\phi_n \geq \phi_{n+1}$ on X and $\lim_{n\to\infty} \phi_n = u$ on X.

The attention is now turned to the subharmonic functions, with its definition and the relationship it has with convex functions.

A subharmonic function is the complex version of a convex function, see [8]. In the context of the complex plane, the connection to the convex function can be obtained by the fact that, a subharmonic function on $D \subset$ that is constant in the imaginary direction, is convex in the real

direction and vice versa. When the switch from convex functions to subharmonic functions is made, $\frac{d^2}{dx^2}$ is replaced by $4\frac{\partial^2}{\partial z \partial \overline{z}}$, where
$\frac{\partial}{\partial z} = \frac{1}{2}(\frac{\partial}{\partial x} + \frac{1}{i}\frac{\partial}{\partial y})$ and $\frac{\partial}{\partial \overline{z}} = \frac{1}{2}(\frac{\partial}{\partial x} - \frac{1}{i}\frac{\partial}{\partial y})$. Let $z = x + iy$ and $\overline{z}$ the conjugate of z, this yields
$4\ \frac{\partial^2}{\partial z \partial \overline{z}} = (\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}) = \Delta($ the Laplacian $)$.

Definition 2.5. [4] A C^2 function φ on a subset of the complex plane is said to be harmonic if and only if, $\Delta\varphi = 0$.

Any harmonic function is C^∞, where C^∞ denotes the set of infinitely continuously differentiable functions.

Definition 2.6. [8] Let D be a domain in and let $h : D \to [-\infty, \infty)$ be an upper semicontinuous function which is not identically - ∞. Then h is said to be *subharmonic*, if any one of the following three equivalent conditions hold:

(a) For all $z_0 \in D$ and $r > 0$ such that the closed ball
$\overline{B_r(z_0)} = \{z \in : |z - z_0| \leq r\} \subset D$, we have $h(z_0) \leq \frac{1}{2\pi}\int_0^{2\pi} h(z_0 + re^{i\theta})d\theta$.

that is, $h(z_0)$ satisfies the local submean inequality.

(b) $4\ \frac{\partial^2 h}{\partial z \partial \overline{z}} \geq 0$, in the sense of distributions, (and in the usual sense if h is a C^2 function).

(c) If H is a relatively compact domain in D and g is harmonic in H and continuous on $\overline{H}$, then $h \leq g$ on $\partial H \Rightarrow h \leq g$ on H, where ∂H is the boundary of H.

It is noticed that h is superharmonic if $-h$ is subharmonic.

The integral in (a) is well defined, since we have seen that, an upper semicontinuous function is locally the limit of a monotone decreasing sequence of continuous functions. So by Lebesgue's Monotone Convergence Theorem, h is then Lebesgue measurable, and

$$\int_0^{2\pi} h(z_0 + re^{i\theta})d\theta = \lim_{n\to\infty} \int_0^{2\pi} \phi_n(z_0 + re^{i\theta})d\theta,$$

where ϕ_n is as in the conclusion of Theorem (2.2)

The condition (c) is known as the *principle of the harmonic majorant.* The maximum principle can be derived for subharmonic functions. (that is, the maximum of a subharmonic function cannot be achieved in the interior of its domain unless the function is constant).

2.3 Some Properties of Subharmonic Functions

Some of the properties of subharmonic functions are given in a form of a theorem as:

Theorem 2.7. [7, 9]

1. If u_1, u_2 and u are subharmonic functions on D and $\beta \geq 0$ a scalar, then $u_1 + u_2, \beta u$ and $\max\{u_1(z), u_2(z)\}$ are subharmonic on D.
2. If (u_n) is a decreasing sequence of subharmonic functions, then $u(z) := \lim_{n\to\infty} u_n(z)$ is subharmonic.
3. If (u_i) is a family of subharmonic functions and if $u(z) := \sup_i u_i(z)$ is upper semicontinuous, then u is subharmonic.

Theorem 2.8. [4] Let u be a subharmonic function on a domain D in , with u not identically $-\infty$ on D. Then u is locally integrable on D , that is

$$\int\int_K |u(x, y)|dxdy < \infty,$$

for each compact subset K of D.

2.4 Some Examples of Subharmonic Functions

1. If h is holomorphic on $D \subseteq^n$, then $|h|^p$ is subharmonic for all $p > 0$.
2. If h is holomorphic on $D \subseteq^n$, then $\log^+ |h|$ is subharmonic, where $\log^+ x = \max\{0, \log x\}, x > 0$.
3. Let $U \subset$ be open and $u : U \to [0, \infty)$. Then $\log u$ is subharmonic on U iff $u|e^p|$ is subharmonic on U for every polynomial p (with complex coefficients).

Lemma 2.9. [2] *Let φ be a superquadratic function with P_x as defined above in Definition 1.1.*

1. *Then* $\varphi(0) \leq 0$.
2. *If* $\varphi(0) = \varphi'(0) = 0,$ *then* $P_x = \varphi'(x)$ *whenever* φ *is differentiable at* $x > 0$.
3. *If* $\varphi \geq 0,$ *then* φ *is convex and* $\varphi(0) = \varphi'(0) = 0$.

Lemma 2.10. [3] Suppose that φ is superquadratic and non-negative. Then φ is convex and increasing. Also if P_x is as in Definition 1.1, then $P_x \geq 0$.

3 Main Results

Proposition 3.1. Let $0 \leq a < b \leq \infty$ and let $u : U \to (a, b)$ be a subharmonic function on an open set $U \in$. Let $\varphi : [0, \infty) \to \mathbb{R}$ be a non-negative superquadratic function. Then $\varphi \ o \ u$ is subharmonic on U.

Proof

By Lemma 2.10, φ is an increasing function on $[0, \infty)$ and we have that the set $\{z \in U : u(z) < \gamma\}$ is open, since u is subharmonic on U and $\gamma \in \mathbb{R}$.

So $\forall z$ satisfying $u(z) < \gamma$, we have $\varphi(u(z)) < \varphi(\gamma)$, thus $\{z \in U : u(z) < \gamma\} = \{z \in U : \varphi(u(z)) < \varphi(\gamma)\}$,

which is open in U and so $\varphi \ o \ u$ is upper semicontinuous.

We next show that $\varphi \ o \ u$ satisfies the submean inequality.(See Definition 2.4(a))

Now

$$(\varphi \ o \ u)(z) \leq \varphi(\tfrac{1}{2\pi}\textstyle\int_0^{2\pi} u(z + re^{i\theta})d\theta), \text{ since } u \text{ is subharmonic and } \varphi \text{ is increasing.}$$

So by the refined Jensen's inequality (1.2) we obtain,

$$\begin{aligned}(\varphi \ o \ u)(z) &\leq \frac{1}{2\pi}\int_0^{2\pi} \varphi(u(z + re^{i\theta}))d\theta(z) \\ &- \frac{1}{2\pi}\int_0^{2\pi} \varphi\big(\big|u(z + re^{i\theta}) - \frac{1}{2\pi}\int_0^{2\pi} u(z + re^{i\theta})d\theta\big|\big)d\theta(z). \end{aligned} \tag{3.1}$$

Since $\varphi \geq 0$, then

$$(\varphi \ o \ u)(z) \leq \tfrac{1}{2\pi}\textstyle\int_0^{2\pi} \varphi(u(z + re^{i\theta}))d\theta(z),$$

which shows that $\varphi \ o \ u$ satisfies the local submean inequality and is thus subharmonic.

The inequality (3.1) is a refinement of the submean inequality.

Proposition 3.2. Let $\vartheta : [0, \infty) \to [0, \infty)$ be an increasing convex function such that $\vartheta(0) = 0$, let $\varphi : [0, \infty) \to [0, \infty)$ be continuously differentiable and $\varphi(0) = 0$ and let $u : U \to (a, b)$ be subharmonic, where $0 \leq a \leq b \leq \infty$ and $U \subset$. If $\varphi^{'}$ is superadditive or $\frac{\varphi^{'}}{Id}$ is non-decreasing, then $(\vartheta\ o\ \varphi\ o\ u)$ is subharmonic.

We first give some preliminary results which will help us in the proof of this proposition.

Definition 3.3. [2] A function $k : [0, \infty) \to \mathbb{R}$ is *superadditive* provided
$k(x + y) \geq k(x) + k(y)$.

Lemma 3.4. [2] *Suppose* $\varphi : [0, \infty) \to \mathbb{R}$ *is continuously differentiable and* $\varphi(0) \leq 0$. *If* $\varphi^{'}$ *is superadditive or* $\frac{\varphi^{'}}{Id}$ *is non-decreasing, then* φ *is superquadratic.*

Lemma 3.5. [2] *Suppose* φ *is differentiable and* $\varphi(0) = \varphi^{'}(0) = 0$. *If* φ *is superquadratic, then* $\frac{\varphi}{(Id)^2}$ *is non-decreasing on* $(0, \infty)$.

Proof of Lemma 3.5

From Lemma 2.9, the constant P_x in the Definition 1.1 is necessarily $\varphi^{'}(x)$.

When we set $y = 0$ in Definition 1.1 we obtain,

$$\varphi(0) - \varphi(x) - \varphi(x) \geq \varphi^{'}(x)(0 - x), \text{ since } x \geq 0$$

and so,

$$x\varphi^{'}(x) \geq 2\varphi(x).$$

Now the first derivative of $\frac{\varphi}{Id^2}$, is

$$\frac{x\varphi^{'}(x) - 2\varphi(x)}{(x^3)} \geq 0,$$

and it follows that $\frac{\varphi(x)}{(x)^2}$ is non-decreasing.

We give a neccessary and sufficient conditions for a function to be superquadratic

Lemma 3.6. Suppose φ is continuosly differentiable and $\varphi(0) = \varphi^{'}(0) = 0$. Then φ is superquadratic if and only if $\frac{\varphi^{'}}{Id}$ is non-decreasing or $\varphi^{'}$ is superadditive on $(0, \infty)$.

Proof

Suppose φ is superquadratic, then from Lemma 2.9, the constant P_x in the Definition 1.1 is necessarily $\varphi^{'}(x)$.

When we set $y = 0$ in Definition 1.1 we obtain,

$$\varphi(0) - \varphi(x) - \varphi(x) \geq \varphi^{'}(x)(0 - x), \text{ since } x \geq 0$$

and so,

$$x\varphi^{'}(x) \geq 2\varphi(x).$$

Dividing through the above expression by x^2, we have

$$\frac{\varphi^{'}(x)}{2x} \geq \frac{\varphi(x)}{x^2}$$

and

$$\frac{\varphi^{'}(x)}{x} \geq \frac{\varphi^{'}(x)}{2x} \geq \frac{\varphi(x)}{x^2}, \forall x \in (0, \infty).$$

But from Lemma 3.5, we have that $\frac{\varphi^{'}(x)}{x^2}$ is non-decreasing on $(0, \infty)$, hence $\frac{\varphi^{'}(x)}{x}$ is non-decreasing on $(0, \infty)$.

By assumption φ is continuously differentiable and superquadratic so for $x \leq y$, we have, by (1.3)

$$\varphi(y) - \varphi(x) - (y - x)\varphi^{'}(x) - \varphi(y - x) \geq 0,$$

the left hand of which is

$$\int_x^y [\varphi^{'}(t) - \varphi^{'}(x) - \varphi^{'}(t - x)]dt,$$

since $\varphi(0) = 0$.

Let $h(t) = \varphi^{'}(t) - \varphi^{'}(x) - \varphi^{'}(t - x)$ and we define a function g given by

$$g(y) = \int_x^y h(t)dt, 0 \leq x \leq y < \infty.$$

Thus $g(y) \geq 0$ and by definition the function h is continuous $\forall x, t \in [0, \infty)$, so by the Fundamental Theorem of Calculus we have that,

$$g^{'}(y) = \varphi^{'}(y) - \varphi^{'}(x) - \varphi^{'}(y - x) \geq 0.$$

Thus for $x \leq y$, we have $\varphi^{'}(y) - \varphi^{'}(x) \geq \varphi^{'}(y - x)$.

Interchanging the roles of x and y in the above discussion we have that,
$\varphi^{'}(x) - \varphi^{'}(y) \geq \varphi^{'}(x - y)$, for $y \leq x$.

Hence $\forall x, y \in (0, \infty), \varphi^{'}(x + y) \geq \varphi^{'}(x) + \varphi^{'}(y)$, So $\varphi^{'}$ is superadditive.

Conversely see [2, lemma 3.1]

If $\varphi^{'}$ is superadditive, then for $x \leq y$ we have,

$$\begin{aligned} 0 &\leq \int_x^y [\varphi^{'}(t) - \varphi^{'}(x) - \varphi^{'}(t - x)]dt \\ &= \varphi(y) - \varphi(x) - (y - x)\varphi^{'}(x) - \varphi(y - x). \end{aligned}$$

For $y \leq x$, we have that

$$\begin{aligned} 0 &\leq \int_y^x [\varphi^{'}(x) - \varphi^{'}(x - t) - \varphi^{'}(t)]dt \\ &= \varphi(y) - \varphi(x) + (x - y)\varphi^{'}(x) - \varphi(x - y). \end{aligned}$$

Thus $\forall x, y \geq 0$ we have that,

$$\varphi(y) \geq \varphi(x) + \varphi^{'}(x)(y - x) + \varphi(|y - x|).$$

Setting $\varphi^{'}(x) = P_x$, we have that φ is superquadratic.

If φ is continuosly differentiable and $\frac{\varphi^{'}(x)}{x}$ is non-decreasing then
$\forall x, y \geq 0$, $\frac{\varphi^{'}(x+y)}{x+y} \geq \frac{\varphi^{'}(x)}{x}$.

Now

$$\begin{aligned}\varphi^{'}(x+y) &= x\frac{\varphi^{'}(x+y)}{x+y} + y\frac{\varphi^{'}(x+y)}{x+y} \\ &\geq \varphi^{'}(x) + \varphi^{'}(y).\end{aligned}$$

Hence $\varphi^{'}$ is superadditive on $(0, \infty)$, which reduces to the first case.

Proof of Proposition 3.2. Since φ satisfies all the conditions in Lemma 3.6, then φ is superquadratic.

From proposition 3.1, $\varphi \ o \ u$ is subharmonic and therefore $(\vartheta \ o \ \varphi \ o \ u)$ is upper semicontinuous, since ϑ is an increasing function.

We have by the refined Jensen's inequality that

$$\begin{aligned}&(\vartheta(\varphi(u)))(z) \\ &\leq \vartheta(\frac{1}{2\pi}\big[\int_0^{2\pi}\varphi(u(z+re^{i\theta})) - \int_0^{2\pi}\varphi\big(\big|u(z+re^{i\theta}) - \frac{1}{2\pi}\int_0^{2\pi}u(z+re^{i\theta})d\theta\big|\big)\big]d\theta(z)),\end{aligned}$$

and

$$\begin{aligned}&\vartheta(\frac{1}{2\pi}\big[\int_0^{2\pi}\varphi(u(z+re^{i\theta})) - \int_0^{2\pi}\varphi\big(\big|u(z+re^{i\theta}) - \frac{1}{2\pi}\int_0^{2\pi}u(z+re^{i\theta})d\theta\big|\big)\big]d\theta(z)) \\ &\leq \vartheta(\frac{1}{2\pi}\int_0^{2\pi}\varphi(u(z+re^{i\theta}))d\theta(z)) - \vartheta(\frac{1}{2\pi}\int_0^{2\pi}\varphi\big(\big|u(z+re^{i\theta}) - \int_0^{2\pi}u(z+re^{i\theta})d\theta\big|\big)d\theta(z)),\end{aligned}$$

since ϑ is superadditive.

From the Jensen's inequality we have that

$$\vartheta(\frac{1}{2\pi}\int_0^{2\pi}\varphi(u(z+re^{i\theta}))d\theta) \leq \frac{1}{2\pi}\int_0^{2\pi}\vartheta[\varphi(u(z+re^{i\theta}))]d\theta,$$

since ϑ is convex.

So

$$\begin{aligned}&(\vartheta(\varphi(u)))(z) \\ &\leq \frac{1}{2\pi}\int_0^{2\pi}\vartheta[\varphi(u(z+re^{i\theta}))]d\theta(z) - \vartheta\big(\frac{1}{2\pi}\int_0^{2\pi}\varphi\big(\big|u(z+re^{i\theta}) - \int_0^{2\pi}u(z+re^{i\theta})d\theta\big|\big)\big)d\theta(z)\end{aligned} \tag{3.2}$$

Now ϑ maps into $[0, \infty)$, that is, $\vartheta\big(\frac{1}{2\pi}\int_0^{2\pi}\varphi\big(\big|u(z+re^{i\theta}) - \int_0^{2\pi}u(z+re^{i\theta})d\theta\big|\big)\big) \geq 0$.
So

$$\vartheta(\varphi(u))(z) \leq \frac{1}{2\pi}\int_0^{2\pi}\vartheta[\varphi(u(z+re^{i\theta}))]d\theta(z).$$

Thus $(\vartheta(\varphi(u)))$ satisfies the local submean inequality, hence subharmonic.

The inequality (3.2) is a refinement of the local submean inequality.

Lemma 3.7. Every positive superquadratic function φ defined on $[0, \infty)$ is superadditive

Proof

From lemma 2.9, we have that φ is convex and $\varphi(0) = 0$, so for all $0 \leq \alpha \leq 1$, $\varphi(\alpha x) \leq \alpha\varphi(x), \forall x \in [0, \infty)$.

Now for any $x, y \in [0, \infty)$

$$\begin{aligned} \varphi(x) + \varphi(y) &= \varphi(\frac{x+y}{x+y}x) + \varphi(\frac{x+y}{x+y}y) \\ &= \varphi(\frac{x}{x+y}(x+y)) + \varphi(\frac{y}{x+y}(x+y)) \\ &\leq \frac{x}{x+y}\varphi(x+y) + \frac{y}{x+y}\varphi(x+y) \\ &= \varphi(x+y), \end{aligned}$$

hence for all $x, y \in [0, \infty)$, φ is superadditive.

4 Conclusion

From the above discussions, one realizes that upon replacing the convex function with a superquadratic function in the various compositions, the local submean inequality is refined, hence giving us a sharper inequality.

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Competing Interests

The authors declare that no competing interests exist.

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