Journal of Advances in Mathematics and Computer Science

35(8): 55-64, 2020; Article no.JAMCS.62524

ISSN: 2456-9968

(Past name: British Journal of Mathematics & Computer Science, Past ISSN: 2231-0851)

Generating Distribution Functions Based on Burr Differential Equation

Reinpeter Ondeyo Momanyi[1*] **and J. A. M. Ottieno**[1]

[1]*School of Mathematics, College of Biological and Physical Sciences (CBPS), University of Nairobi, Kenya.*

Authors' contributions

This work was carried out in collaboration between both authors. ROM wrote the first draft of the manuscript. JAMO managed the supervision of the manuscript. Both authors read and approved the final manuscript.

Article Information

DOI: 10.9734/JAMCS/2020/v35i830313

Editor(s):

(1) Dr. Dragoş - Pătru Covei, The Bucharest University of Economic Studies, Romana.

(2) Dr. Francisco Welington de Sousa Lima, Universidade Federal do Piauí, Brazil.

(3) Dr. Raducanu Razvan, Al. I. Cuza University, Romania.

Reviewers:

(1) Prasanta Kumar Das, KIIT University, India.

(2) Oyelami Funmilayo Helen, Afe Babalola University, Nigeria.

Complete Peer review History: http://www.sdiarticle4.com/review-history/62524

Received: 02 September 2020

Accepted: 09 November 2020

Mini-review Article

Published: 02 December 2020

Abstract

One of the most prominent families of statistical distributions is the Burr's system. Recent renewed interest in developing more flexible statistical distributions led to the re-examination of Burr's system. Solutions of Burr differential equation are expressed in terms of distribution functions. Burr [1] considered only 12 distribution functions known in literature as the Burr system of distributions, yet there are more than that in number. Studying the Burr system, it was realized that 9 of the Burr distributions are powers of $cdf's$, popularly now known as exponentiated distributions. The remaining 3 are direct solutions in terms of $cdf's$.

Detailed studies using generator approach techniques to generate Burr distributions has not been undertaken in literature. This motivated us to generalize solutions of Burr differential equation by generator approach. With this aim in mind, beta generator method, exponentiated generator method and beta-exponentiated generator method (a combination of beta and exponentiated generator methods) was proposed. However in this paper, we will focus on exponentiated generator technique as it generates $cdf's$. The other two generator approach techniques generate $pdf's$ and distributions of order statistics.

*Corresponding author: E-mail: reinp.mom@gmail.com;

Keywords: Cumulative distribution function (cdf), $F(x)$; probability density function (pdf), $f(x)$; beta generator; exponentiated generator; beta-exponentiated generator; income elasticity; reverse hazard function; hazard function; order statistics.

2010 Mathematics Subject Classification: 53C25, 83C05, 57N16.

1 Introduction

Kotz and Vicari [2] highlighted methods developed before 1980s focused on systems of frequency functions which started with the differential equation approach. This approach was significant and led to the construction of Pearsons system (Pearson[3]) and The Burr system (Burr[1]). After 1980s, methods of generating new distributions shifted to adding parameters to an existing distribution and combining existing distributions into new distributions.

Burr [1] introduced a system of distributions by considering distribution functions $F(x)$ satisfying the differential equation of the form

$$y' = y(1-y)g(x,y) \tag{1.1}$$

where $y' = \dfrac{dy}{dx} = \dfrac{dF(x)}{dx} = f(x)$, $y = F(x)$ and $g(x,y)$ is a non negative function for $0 \le y \le 1$ and x in the range over which the solution is to be used.

Burr gave the following twelve solutions in table 1 (usually referred to by number). The Roman numeral description for the 12 types was first used by Johnson and Kotz [4].

Table 1. The Burr system of distributions

Type	F(x)	Support
I	x	$0 < x < 1$
II	$\left[e^{-x}+1\right]^{-r}$	$-\infty < x < \infty$
III	$\left[x^{-c}+1\right]^{-r}$	$0 < x < \infty$
IV	$\left[\left(\frac{c-x}{x}\right)^{\frac{1}{c}}+1\right]^{-r}$	$0 < x < c$
V	$\left[ke^{-\tan x}+1\right]^{-r}$	$-\frac{\pi}{2} < x < \frac{\pi}{2}$
VI	$\left[ke^{-c\sinh x}+1\right]^{-r}$	$-\infty < x < \infty$
VII	$2^{-r}\left[1+\tanh x\right]^{r}$	$-\infty < x < \infty$
VIII	$\left[\frac{2}{\pi}\arctan e^{x}\right]^{r}$	$-\infty < x < \infty$
IX	$1-\frac{2}{c\left[(1+e^{x})^{k}-1\right]+2}$	$-\infty < x < \infty$
X	$\left[1-e^{-x^{2}}\right]^{r}$	$0 < x < \infty$
XI	$\left[x-\frac{1}{2\pi}\sin 2\pi x\right]^{r}$	$0 < x < 1$
XII	$1-\left[1+x^{c}\right]^{-k}$	$0 < x < \infty$

where c, k and r are positive real numbers.

2 Direct Solution Approach

To obtain the various distribution functions $F(x)$, we will solve the Burr differential equation for the 5 cases of $g(x, y)$.

The cases to be considered are:

Case I: $g(x, y) = \dfrac{g(x)}{y(1-y)}$

Case II: $g(x, y) = g(x)$

Case III: $g(x, y) = \dfrac{g(x)}{xy}$

Case IV: $g(x, y) = \dfrac{r(x)}{(1-y)}$

Case V: $g(x, y) = \dfrac{\mu(x)}{y}$

2.1 Distributions based on Case I of Burr Differential Equation

This is the case when $g(x, y) = \dfrac{g(x)}{y(1-y)}$.

Therefore (1.1) becomes $y' = g(x)$. Solving this we get,

$$F(x) = \int g(x)dx \tag{2.1}$$

2.2 Distributions based on Case II of Burr Differential Equation

This is the case when $g(x, y) = g(x)$.

Therefore (1.1) becomes $y' = y(1-y)g(x)$. Solving this we get,

$$F(x) = \left[e^{-\int g(x)dx} + 1\right]^{-1} \tag{2.2}$$

which was Burr's assumption.

2.3 Distributions based on Case III of Burr Differential Equation

Stoppa [5] proposed a differential equation for income elasticity as

$$\eta(x, F(x)) = \frac{1 - [F(x)]^{\frac{1}{\theta}}}{[F(x)]^{\frac{1}{\theta}}} g(x, F(x)), \qquad x > x_0 \geqslant 0 \tag{2.3}$$

where $\eta(x, F(x)) = x\dfrac{F'(x)}{F(x)}$, *is the* **income elasticity** and $F(x)$ *is the cdf.*

Let $g(x, F(x)) = g(x)$ and $\theta = 1$, then (2.3) becomes $x\dfrac{F'(x)}{F(x)} = \dfrac{1-F(x)}{F(x)}g(x)$ which can be written as $x\dfrac{y'}{y} = \dfrac{1-y}{y}g(x)$.

Re-arranging, $y' = y(1-y)\dfrac{g(x)}{xy}$ which is (1.1) with $g(x, y) = \dfrac{g(x)}{xy}$

According to Kleiber and Kotz [6], Dagum's differential equation is of the form

$$\frac{d\log[F(x) - \delta]}{d\log x} = \theta(x)\phi(F) \leq k, \quad 0 \leq x_0 < x < \infty \tag{2.4}$$

where $k > 0,\ \theta(x) > 0,\ \phi(F) > 0,\ \delta < 1$ and $\dfrac{d\{\theta(x)\phi(F)\}}{dx} < 0.$

When $\delta = 0,\ \theta(x) = g(x)$ and $\phi(F) = \dfrac{1-F}{F}$, (2.4) becomes $\dfrac{d\log F(x)}{d\log x} = g(x)\left[\dfrac{1-F}{F}\right]$

But $\dfrac{d\log F(x)}{d\log x} = x\dfrac{F'(x)}{F(x)}$.

Therefore $x\dfrac{F'(x)}{F(x)} = g(x)\left[\dfrac{1-F}{F}\right]$ which can be written as $x\dfrac{y'}{y} = \dfrac{1-y}{y}g(x)$ and re-arranged to $y' = y(1-y)\dfrac{g(x)}{xy}$. This is the same result as before.

Thus this Burr's equation reduces to $y' = (1-y)\dfrac{g(x)}{x}$. Solving this we get,

$$F(x) = 1 - \exp\left[-\int \frac{g(x)}{x}dx\right] \tag{2.5}$$

2.4 Distributions based on case IV of Burr differential equation

Olapade [7] states that one of the properties of Type I Generalized Logistic distribution is that it satisfies the homogeneous differential equation $(1-e^{-x})F' - be^{-x}F = 0$ where $F' = \dfrac{be^{-x}}{1+e^{-x}}F$.

Therefore $F' = F(1-F)\left(\dfrac{be^{-x}}{1+e^{-x}}\right)\dfrac{1}{(1-F)}$ which can be written as $y' = y(1-y)\left(\dfrac{be^{-x}}{1+e^{-x}}\right)\dfrac{1}{(1-y)}$

Let $g(x) = \left(\dfrac{be^{-x}}{1+e^{-x}}\right)$, thus

$$\begin{aligned} y' &= y(1-y)\frac{g(x)}{(1-y)} \\ y' &= yg(x) \\ g(x) = \frac{y'}{y} &= \frac{f(x)}{F(x)} = r(x) \end{aligned}$$

where $r(x)$ is the **reverse hazard function**.

Re-writing the equation, $y' = y(1-y)\dfrac{r(x)}{(1-y)}$ which is (1.1) with $g(x,y) = \dfrac{r(x)}{(1-y)}$.

Solving this we get,

$$F(x) = \exp\left[\int r(x)dx\right] \tag{2.6}$$

2.5 Distributions based on case V of Burr differential equation

The statistical theory of survival analysis deals with survival time T which is regarded as a continous random variable. Accordingly, the survival time t is a realization of T.

Since T is a continous random variable, the probability of dying at any given time is 0. T has an associated probability density function $f(t)$ and can be characterized in terms of two other functions, namely the **survival function** $S(t)$ and the **hazard function** $h(t)$.

A nonzero probability is obtained only when we consider the probability of dying in an interval of time. Thus $S(t)$, $f(t)$ and $h(t)$ are defined as follows:

$S(t)$ = the probability of an individual alive at time 0 surviving until (at least) time t

$$S(t) = 1 - F(t) \tag{2.7}$$

where $F(t)$ is the *cdf*.

$f(t)$ = the instantaneous probability per unit time than an individual alive at time 0 will die at time t

$$f(t) = -\frac{dS(t)}{dt} \tag{2.8}$$

$h(t)$ = the instantaneous probability per unit time than an individual alive at time t will die in the next instant

$$h(t) = \frac{f(t)}{1 - F(t)} \tag{2.9}$$

It follows that from (2.9), $h(t) = -\frac{d[\log S(t)]}{dt}$. Hence, $h(t)dt = -d[\log S(t)]$

Integrating both sides

$$-\int_0^x h(t)dt = \int_0^x d[\log S(t)]$$

$$\log S(x) - \log S(0) = -\int_0^x h(t)dt$$

$$\log\left(\frac{S(x)}{S(0)}\right) = -\int_0^x h(t)dt$$

But by definition $S(0) = prob\ (T \geq 0) = \int_0^\infty f(x)dx = 1$

Therefore, $\log S(x) = -\int_0^x h(t)dt$. Exponentiating both sides, $S(x) = \exp\left[-\int_0^x h(t)dt\right]$

But $S(x) = 1 - F(x)$, Therefore $1 - F(x) = \exp\left[-\int_0^x h(t)dt\right]$.

Re-arranging,

$$F(x) = 1 - \exp\left[-\int_0^x h(t)dt\right] \tag{2.10}$$

From stochastic approach, Chiang [8] derived the hazard function as follows:

Let $\mu(x)\Delta x + o(\Delta x)$ = *the probability of dying between age x and* $x + \Delta x$ and *prob* $(X \leq x)$ = *the probability of dying at or before age* $x = F(x)$

Then

$$\begin{aligned} F(x+\Delta x) &= \textit{the probability of dying at or before age } x+\Delta x \\ &= \textit{the probability of dying at or before age } x \textit{ or probability} \\ &\quad \textit{of living upto age } x \textit{ and dying between age } x \textit{ and } x+\Delta x \\ &= F(x) + [1-F(x)][\mu(x)\Delta x + o(\Delta x)] \end{aligned}$$

which becomes $F(x + \Delta x) - F(x) = [1 - F(x)][\mu(x)\Delta x + o(\Delta x)]$

Hence

$$\lim_{\Delta x \to 0} \frac{F(x + \Delta x) - F(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{[1 - F(x)][\mu(x)\Delta x + o(\Delta x)]}{\Delta x} \tag{2.11}$$

Since $\lim_{\Delta x \to 0} \frac{o(\Delta x)}{\Delta x} = 0$, (2.12) reduces to

$$f(x) = [1 - F(x)]\mu(x) \tag{2.12}$$

Re-arranging (2.12), we get $\mu(x) = \frac{f(x)}{1 - F(x)}$ which can be written as

$$\mu(x) = \frac{y'}{1 - y} \tag{2.13}$$

Re-writing (2.13) we get $y' = y(1 - y)\frac{\mu(x)}{y}$ which is (1.1) with $g(x, y) = \frac{\mu(x)}{y}$

In demography, the hazard function $h(t)$ is called the **force of mortality** denoted by $\mu(t)$. This implies that (2.13) becomes

$$\int \frac{dy}{(1 - y)} = \int_0^x \mu(t)dt \tag{2.14}$$

Solving (2.14), we get

$$F(x) = 1 - \exp\left[-\int_0^x \mu(t)dt\right] \tag{2.15}$$

same result as (2.10).

3 Exponentiated Generator Approach

This method generates the exponentiated family of distributions. It involves adding extra parameters to an existing distribution and can be applied to any generated family of distributions.

The pioneering work on the exponentiated method is given in Mudholkar and Srivastava [9] when they developed the exponentiated Weibull distribution for modeling bathtub failure-rate data.

Let

$$G(x) = [F(x)]^r, \quad r > 0 \tag{3.1}$$

where $F(x)$ is the old/parent *cdf* and $G(x)$ is the new *cdf*. Then $G(x)$ is an exponentiated distribution.

4 Distribution Functions Obtained

A summary of $cdf's$ obtained from the five cases and exponentiated generator technique is given in table 2.

Table 2. Generated distributions based on Burr differential equation: $y' = y(1-y)g(x,y)$

$g(x), r(x), \mu(t)$	$F(x)$	$[F(x)]^r, \quad r > 0$	**Support**
Case I: $g(x,y) = \frac{g(x)}{y(1-y)}$			
1	x **Standard Uniform (Burr I)**	x^r **Exponentiated Standard Uniform (Burr I)**	$0 < x < 1$
Case II: $g(x,y) = g(x)$			
1	$\left[e^{-x}+1\right]^{-1}$ **Logistic**	$\left[e^{-x}+1\right]^{-r}$ **Type I Generalized Logistic (Burr II)**	$-\infty < x < \infty$
$\frac{c}{x}$	$\left[x^{-c}+1\right]^{-1}$ **Log-Logistic/Fisk**	$\left[x^{-c}+1\right]^{-r}$ **Exponentiated Log-Logistic/Fisk (Burr III)**	$0 < x < \infty$
$[(c-x)x]^{-1}$	$\left[\left(\frac{c-x}{x}\right)^{\frac{1}{c}}+1\right]^{-1}$	$\left[\left(\frac{c-x}{x}\right)^{\frac{1}{c}}+1\right]^{-r}$ **Burr IV**	$0 < x < c$
$\sec^2 x$	$\left[ke^{-\tan x}+1\right]^{-1}$	$\left[ke^{-\tan x}+1\right]^{-r}$ **Burr V**	$-\frac{\pi}{2} < x < \frac{\pi}{2}$
$c\cosh x$	$\left[ke^{-c\sinh x}+1\right]^{-1}$	$\left[ke^{-c\sinh x}+1\right]^{-r}$ **Burr VI**	$-\infty < x < \infty$
2	$2^{-1}\left[1+\tanh x\right]$	$2^{-r}\left[1+\tanh x\right]^r$ **Burr VII**	$-\infty < x < \infty$
Case III: $g(x,y) = \frac{g(x)}{xy}$			
$\frac{ckxe^x(1+e^x)^{k-1}}{c\left[(1+e^x)^k-1\right]+2}$	$1-\frac{2}{c\left[(1+e^x)^k-1\right]+2}$ **Burr IX**	$\left\{1-\frac{2}{c\left[(1+e^x)^k-1\right]+2}\right\}^r$ **Exponentiated Burr IX**	$-\infty < x < \infty$
$2x^2$	$1-e^{-x^2}$	$\left[1-e^{-x^2}\right]^r$ **Burr X**	$0 < x < \infty$
$\frac{ckx^c}{1+x^c}$	$1-\left[1+x^c\right]^{-k}$ **Burr XII**	$\left\{1-\left[1+x^c\right]^{-k}\right\}^r$ **Exponentiated Burr XII**	$0 < x < \infty$
$\frac{cx}{1-cx}$	cx	$[cx]^r$	$0 < x < \frac{1}{c}$
cx	$1-e^{-cx}$	$\left[1-e^{-cx}\right]^r$	$0 < x < \infty$
$(c-x)^{-1}$	$1-\left(\frac{c-x}{x}\right)^{\frac{1}{c}}$	$\left[1-\left(\frac{c-x}{x}\right)^{\frac{1}{c}}\right]^r$	$\frac{c}{2} < x < c$
$cx\sec^2 x$	$1-e^{-c\tan x}$	$\left[1-e^{-c\tan x}\right]^r$	$-\frac{\pi}{2} < x < \frac{\pi}{2}$
α	$1-\left[\frac{1}{x}\right]^{\alpha}$ **Pareto (Type I)**	$\left\{1-\left[\frac{1}{x}\right]^{\alpha}\right\}^r$ **Exponentiated Pareto (Type I)**	$1 < x < \infty$
$\frac{\alpha x}{1+x}$	$1-\left[\frac{1}{1+x}\right]^{\alpha}$ **Pareto (Type II)**	$\left\{1-\left[\frac{1}{1+x}\right]^{\alpha}\right\}^r$ **Exponentiated Pareto (Type II)**	$0 < x < \infty$

$g(x), r(x), \mu(t)$	$F(x)$	$[F(x)]^r, \quad r>0$	**Support**
$\beta x + \dfrac{\alpha x}{1+x}$	$1-\dfrac{e^{-\beta x}}{[1+x]^{\alpha}}$ **Pareto (Type III)**	$\left\{1-\dfrac{e^{-\beta x}}{[1+x]^{\alpha}}\right\}^r$ **Exponentiated Pareto (Type III)**	$0<x<\infty$
$\dfrac{\alpha x(x-\mu)^{\frac{1}{\beta}-1}}{\beta[1+(x-\mu)^{\frac{1}{\beta}}]}$	$1-\left[1+(x-\mu)^{\frac{1}{\beta}}\right]^{-\alpha}$ **Pareto (Type IV)**	$\left\{1-\left[1+(x-\mu)^{\frac{1}{\beta}}\right]^{-\alpha}\right\}^r$ **Exponentiated Pareto (Type IV)**	$\mu<x<\infty$
$\alpha\left(\dfrac{x}{\beta}\right)^{\alpha}$	$1-e^{-\left(\frac{x}{\beta}\right)^{\alpha}}$ **Weibull**	$\left[1-e^{-\left(\frac{x}{\beta}\right)^{\alpha}}\right]^r$ **Exponentiated Weibull**	$0<x<\infty$
$\dfrac{x}{\beta}$	$1-e^{-\frac{x}{\beta}}$ **Exponential**	$\left[1-e^{-\frac{x}{\beta}}\right]^r$ **Exponentiated Exponential**	$0<x<\infty$
$\alpha+2\beta\log x$	$1-e^{-\alpha\log x-\beta(\log x)^2}$ **Benini**	$\left[1-e^{-\alpha\log x-\beta(\log x)^2}\right]^r$ **Exponentiated Benini**	$1<x<\infty$
Case IV: $g(x,y)=\dfrac{r(x)}{(1-y)}$			
$\dfrac{1}{2\cosh x\arctan e^x}$	$\dfrac{2}{\pi}\arctan e^x$	$\left[\dfrac{2}{\pi}\arctan e^x\right]^r$ **Burr VIII**	$-\infty<x<\infty$
$\dfrac{1-\cos 2\pi x}{x-\frac{1}{2\pi}\sin 2\pi x}$	$x-\dfrac{1}{2\pi}\sin 2\pi x$	$\left[x-\dfrac{1}{2\pi}\sin 2\pi x\right]^r$ **Burr XI**	$0<x<1$
$\dfrac{\alpha e^{-x}}{1+e^{-x}}$	$[1+e^{-x}]^{-\alpha}$ **Type I Generalized Logistic (Burr II)**	$[1+e^{-x}]^{-\alpha r}$ **Exponentiated-Type I Generalized Logistic**	$-\infty<x<\infty$
Case V: $g(x,y)=\dfrac{\int_0^x \mu(t)dt}{y}$			
$\dfrac{\alpha}{\beta+t}$	$1-\left(1+\frac{x}{\beta}\right)^{-\alpha}$ **Lomax**	$\left[1-\left(1+\frac{x}{\beta}\right)^{-\alpha}\right]^r$ **Exponentiated Lomax**	$0<x<\infty$
c	$1-e^{-cx}$ **Exponential**	$\left[1-e^{-cx}\right]^r$ **Exponentiated Exponential**	$0<x<\infty$
$c\alpha t^{\alpha-1}$	$1-e^{-cx^{\alpha}}$ **Weibull**	$\left[1-e^{-cx^{\alpha}}\right]^r$ **Exponentiated Weibull**	$0<x<\infty$
$\alpha+\beta t$	$1-e^{-\alpha x-\frac{\beta x^2}{2}}$ **Linear Exponential**	$\left[1-e^{-\alpha x-\frac{\beta x^2}{2}}\right]^r$ **Exponentiated-Linear Exponential**	$0<x<\infty$
$A\log t$	$1-\left(\dfrac{e}{x}\right)^{Ax}$	$\left[1-\left(\dfrac{e}{x}\right)^{Ax}\right]^r$	
$\dfrac{p}{1+kt}$	$1-(1+kx)^{-\frac{p}{k}}$	$\left[1-(1+kx)^{-\frac{p}{k}}\right]^r$	$0<x<\infty$
$\dfrac{1}{\omega-t}$	$\dfrac{x}{\omega}$	$\left[\dfrac{x}{\omega}\right]^r$	$0<x<\omega$
e^{kt-d}	$1-\exp\left\{-\frac{e^{-d}}{k}\left[e^{kx}-1\right]\right\}$ **Gompertz**	$\left(1-\exp\left\{-\frac{e^{-d}}{k}\left[e^{kx}-1\right]\right\}\right)^r$ **Exponentiated Gompertz**	$0<x<\infty$
$A+e^{kt-d}$	$F(x)=1-\exp\left\{-Ax-\frac{e^{-d}}{k}\left[e^{kx}-1\right]\right\}$ **Gompertz-Makeham** $[F(x)]^r=\left(1-\exp\left\{-Ax-\frac{e^{-d}}{k}\left[e^{kx}-1\right]\right\}\right)^r$ **Exponentiated Gompertz-Makeham**		$0<x<\infty$

$g(x), r(x), \mu(t)$	$F(x)$	$[F(x)]^r, \quad r>0$	**Support**
$A + Ht + e^{kt-d}$	$F(x) = 1 - \exp\left\{-Ax - \frac{Hx^2}{2} - \frac{e^{-d}}{k}\left[e^{kx}-1\right]\right\}$	$[F(x)]^r = \left(1 - \exp\left\{-Ax - \frac{Hx^2}{2} - \frac{e^{-d}}{k}\left[e^{kx}-1\right]\right\}\right)^r$	$0 < x < \infty$
$A + e^{kt-d} + \frac{D}{N-t}$	$F(x) = 1 - \left[\frac{N}{N-x}\right]^{-D} \exp\left\{-Ax - \frac{e^{-d}}{k}\left[e^{kx}-1\right]\right\}$	$[F(x)]^r = \left(1 - \left[\frac{N}{N-x}\right]^{-D} \exp\left\{-Ax - \frac{e^{-d}}{k}\left[e^{kx}-1\right]\right\}\right)^r$	
$A + Bc^t + Mn^t$	$F(x) = 1 - \exp\left\{-Ax - \frac{B}{\log c}\left[c^x-1\right] - \frac{M}{\log n}\left[n^x-1\right]\right\}$	$[F(x)]^r = \left(1 - \exp\left\{-Ax - \frac{B}{\log c}\left[c^x-1\right] - \frac{M}{\log n}\left[n^x-1\right]\right\}\right)^r$	$0 < x < \infty$
$\frac{a}{\sqrt{t}} + b + ct^{\frac{1}{3}}$	$F(x) = 1 - \exp\left\{-2a\sqrt{x} - bx - \frac{3}{4}c\left(x^{\frac{1}{3}}\right)^4\right\}$	$[F(x)]^r = \left[1 - \exp\left\{-2a\sqrt{x} - bx - \frac{3}{4}c\left(x^{\frac{1}{3}}\right)^4\right\}\right]^r$	$0 < x < \infty$
$\frac{c+2dt}{1-ct-dt^2}$	$F(x) = cx + dx^2$ $[F(x)]^r = \left[cx + dx^2\right]^r$	$\frac{-c-\sqrt{c^2+4d}}{2d} < x < \frac{-c+\sqrt{c^2+4d}}{2d}$	
$\frac{A+Bc^t}{1+Dc^t}$	$F(x) = 1 - \left[\frac{(1+c^xD)^{\frac{1}{c^x}}}{1+D}\right]^{-\frac{1}{D}} \exp\left\{-\frac{B}{D}x\right\}$	$[F(x)]^r = \left(1 - \left[\frac{(1+c^xD)^{\frac{1}{c^x}}}{1+D}\right]^{-\frac{1}{D}} \exp\left\{-\frac{B}{D}x\right\}\right)^r$	
$\frac{A+Bc^t}{1+Dc^t} + Ec^t$	$F(x) = 1 - \left[\frac{(1+c^xD)^{\frac{1}{c^x}}}{1+D}\right]^{-\frac{1}{D}} \exp\left\{-\frac{B}{D}x - \frac{E}{\log c}\left[c^x-1\right]\right\}$	$[F(x)]^r = \left(1 - \left[\frac{(1+c^xD)^{\frac{1}{c^x}}}{1+D}\right]^{-\frac{1}{D}} \exp\left\{-\frac{B}{D}x - \frac{E}{\log c}\left[c^x-1\right]\right\}\right)^r$	
$\mu(t) = a_1e^{-b_1t} + a_2e^{\frac{-b_2}{2}(t-c)^2} + a_3e^{b_3t}$	$F(x) = 1 - \exp\left\{\frac{a_1}{b_1}\left[e^{-b_1x}-1\right] + \frac{a_2}{b_2}\left[\frac{e^{\frac{-b_2c^2}{2}}\left\{ce^{\frac{-b_2(x^2-2xc)}{2}} + (x-c)\right\}}{c(x-c)}\right] - \frac{a_3}{b_3}\left[e^{b_3x}-1\right]\right\}$	$[F(x)]^r = \left(1 - \exp\left\{\frac{a_1}{b_1}\left[e^{-b_1x}-1\right] + \frac{a_2}{b_2}\left[\frac{e^{\frac{-b_2c^2}{2}}\left\{ce^{\frac{-b_2(x^2-2xc)}{2}} + (x-c)\right\}}{c(x-c)}\right] - \frac{a_3}{b_3}\left[e^{b_3x}-1\right]\right\}\right)^r$	$0 < x < \infty$
$a_1e^{-b_1t} + a_2 + a_3e^{b_3t}$	$F(x) = 1 - \exp\left\{\frac{a_1}{b_1}\left[e^{-b_1x}-1\right] - a_2x - \frac{a_3}{b_3}\left[e^{b_3x}-1\right]\right\}$	$[F(x)]^r = \left(1 - \exp\left\{\frac{a_1}{b_1}\left[e^{-b_1x}-1\right] - a_2x - \frac{a_3}{b_3}\left[e^{b_3x}-1\right]\right\}\right)^r$	$0 < x < \infty$
$\mu(t) = \frac{\frac{2a}{t^2}\left[\frac{a-t}{t}\right] + \frac{b}{c}\left[-e^{-\frac{t}{c}} + e^{-\frac{t}{d}}\right]}{\left[\frac{a-t}{t}\right]^2 - b\left[e^{-\frac{t}{c}} - e^{-\frac{t}{d}}\right]}$	$F(x) = 1 - \left(\left[\frac{a-x}{x}\right]^2 - b\left[e^{-\frac{x}{c}} - e^{-\frac{x}{d}}\right]\right)$	$[F(x)]^r = \left\{1 - \left(\left[\frac{a-x}{x}\right]^2 - b\left[e^{-\frac{x}{c}} - e^{-\frac{x}{d}}\right]\right)\right\}^r$	

5 Conclusions

a In the section (2) and subsections (2.1), (2.2), (2.3), (2.4), (2.5), we have highlighted and solved the five cases for constructing distribution functions based on Burr differential equation.

b In the section (3), we have further examined the use of exponentiated generator approach to construct distribution functions by adding an extra parameter r.

c In the section (4), 86 distribution functions have been introduced which is more than the 12 proposed by Burr [1]. Extensive study on reverse hazard functions is required as this will facilitate other researchers to construct more distribution functions in future.

Acknowledgement

The authors are grateful to the editor and anonymous reviewers for their constructive comments and valuable suggestions which certainly improved the quality of the paper.

Competing Interests

Authors have declared that no competing interests exist.

References

[1] Burr IW. Cumulative frequency functions. Annals of Mathematical Statistics. 1942;13:215-232.

[2] Kotz S, Vicari D. Survey of developments in the theory of continuous skewed distributions. Metron. 2005;LXIII:225-261.

[3] Pearson K. Contributions to the mathematical theory of evolution. II. Skew variation in homogeneous material. Transactions of the Royal Society London. 1895;A186:343-415.

[4] Johnson NL, Kotz S. Continuous univariate distributions. Houghton Mifflin, Boston. 1970;1.

[5] Stoppa G. A new generating system of income distribution models. Quaderni di statistica e matematica applicata alle scienze economico-sociali. 1990a;12:47-55.

[6] Kleiber C, Kotz S. Statistical size distributions in economics and actuarial sciences. John Wiley & Sons, Hoboken, New Jersey; 2003.

[7] Olapade AK. Some properties of the type I generalized logistic distribution. Intro Stats, 2; 2000.

[8] Chiang Ch. L. Introduction to stochastic prcoesses in biostatistics. Wiley & Sons, Inc., New York, London, Sydney; 1968. 301 S, 1 Abb, 25 Tab, Preis 131 s.

[9] Mudholkar GS, Srivastava DK. Exponentiated Weibull family for analyzing bathtub failure-rate data. IEEE Transactions on Reliability. 1993;42:299-302.

Peer-review history:
The peer review history for this paper can be accessed here (Please copy paste the total link in your browser address bar)
http://www.sdiarticle4.com/review-history/62524