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Further Results on Unicity of *q*- Shift Difference-Differential Polynomials of Meromorphic Functions

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Author's contributions

This work was carried out in collaboration between both authors. Both authors read and approved the final manuscript.

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Abstract

This article is devoted to studying the uniqueness of q-shift difference-differential polynomials of transcendental meromorphic functions of zero order with weight l. Here we established unicity results on difference operators when two such functions sharing 1- points. Our findings extend some previous existing results.

Keywords: Meromorphic functions; Shift; Sharing values; uniqueness; differential polynomial; zero order.

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1 Introduction and Definitions

The Value Distribution Theory of Nevanlinna is about a century old and still is an active area of research. It has a wide range of applications within and outside function theory. In this paper, by meromorphic functions we will always mean meromorphic functions in the complex plane. We assume that the reader is familiar with the classical value distribution theory of meromorphic functions as described in say, the standard monograph(see [1-3] and also for the elementary definitions and standard notations of the Nevanlinna value distribution theory such as $T(r,\mathfrak{f}),\ N(r,\mathfrak{f}),\ N\left(r,\frac{1}{\mathfrak{f}}\right),\ m(r,\mathfrak{f})$ etc.

For a nonconstant meromorphic function $\mathfrak{f}$, we denote by $T(r,\mathfrak{f})$ the Nevanlinna characteristic of $\mathfrak{f}$ and by $S(r,\mathfrak{f})$ any quantity satisfying $S(r,\mathfrak{f}) = o\{T(r,\mathfrak{f})\}$ as $r \to \infty$ outside of a possible exceptional set E of finite linear measure. The meromorphic function $a(z)$ is called a small function of $\mathfrak{f}$ if $T(r,a) = S(r,\ \mathfrak{f})$. Two non-constant meromorphic functions $\mathfrak{f}$ and $\mathfrak{g}$ share a small function a CM (counting multiplicities) provided that $\mathfrak{f} - a$ and $\mathfrak{g} - a$ have the same set of zeros with the same multiplicities $\mathfrak{f}$ and $\mathfrak{g}$ share a IM (ignoring multiplicities) if we do not consider the multiplicities. Set $E(a,\mathfrak{f}) = \{z\colon\ \mathfrak{f}(z) - a = 0\}$, where a zero point with multiplicity m is counted m times in the set. If these zeros are only counted once, then we denote the set by $\bar{E}(a,\mathfrak{f})$. If $E(a,\mathfrak{f}) = E(a,\mathfrak{g})$, then we say that $\mathfrak{f}$ and $\mathfrak{g}$ share the value a CM; if $\bar{E}(a,\mathfrak{f}) = \bar{E}(a,\mathfrak{g})$, then we say that $\mathfrak{f}$ and $\mathfrak{g}$ share the value a IM.

Definition 1.1[4] Let $k \in \mathbb{N} \cup \{0\} \cup \{\infty\}$ and $a \in \mathbb{C} \cup \{\infty\}$. We denote by $E_k(a;\mathfrak{f})$ the set of all a points of f where an a- point of multiplicity m is counted m times if $m \le k$ and $k+1$ times if $m > k$. If $E_k(a;\mathfrak{f}) = E_k(a;\mathfrak{g})$, we say that $\mathfrak{f}$, $\mathfrak{g}$ share the value a with weight k.

Definition 1.2.[4,5] Let $\mathfrak{f}$, $\mathfrak{g}$ share a value $(a,0)$. We denote by $\bar{N}_*(r,a;\mathfrak{f},\mathfrak{g})$ the reduced counting function of those a-points of $\mathfrak{f}$ whose multiplicities differ from the multiplicities of the corresponding a-points of $\mathfrak{g}$.

Clearly $\bar{N}_*(r,a;\mathfrak{f},\mathfrak{g}) = \bar{N}_*(r,a;\mathfrak{g},\mathfrak{f})$ and $\bar{N}_*(r,a;\mathfrak{f},\mathfrak{g}) = \bar{N}_L(r,a;\mathfrak{f}) + \bar{N}_L(r,a;\mathfrak{g})$.

Definition 1.3.[6] Let $a \in \mathbb{C} \cup \{\infty\}$. We denote by $N(r,a;\mathfrak{f}| = 1)$ the counting function of simple a points of $\mathfrak{f}$. For $p \in \mathbb{N}$ we denote by $N(r,a;\mathfrak{f}| \le p)$ the counting function of those a points of $\mathfrak{f}$ whose multiplicities are not greater than p. By $\bar{N}(r,a;\mathfrak{f}| \le p)$ we denote the corresponding reduced counting function.

Definition 1.4.[4] Let $p \in \mathbb{N} \cup \{\infty\}$. We denote by $N_p(r,a;\mathfrak{f})$ the counting function of a- points of $\mathfrak{f}$, where an a-point of multiplicity m is counted m times if $m \le p$ and p times if $m > p$. Then $N_p(r,a;\mathfrak{f}) = \bar{N}(r,a;\mathfrak{f}) + \bar{N}(r,a;\mathfrak{f}| \ge 2) + \cdots + \bar{N}(r,a;\mathfrak{f}| \ge p)$.Clearly $N_1(r,a;\mathfrak{f}) = \bar{N}(r,a;\mathfrak{f})$.

Definition 1.5.[7] Let $\mathfrak{f},\mathfrak{g}$ be two non-constant meromorphic functions such that $\mathfrak{f}$ and $\mathfrak{g}$ share $(a.0)$. Let z_0 be an a- point of $\mathfrak{f}$ with multiplicity p, an a- point of $\mathfrak{g}$ with multiplicity q. We denote by $\bar{N}_L(r,a;\mathfrak{f})$ the reduced counting function of those a-points of $\mathfrak{f}$ and $\mathfrak{g}$ where $p > q$, by $N_E^{1)}(r,a;\mathfrak{f})$ the counting function of those a-points of $\mathfrak{f}$ and $\mathfrak{g}$ where $p = q = 1$, by $N_E^{(2}(r,a;\mathfrak{f})$ the reduced counting function of those a- points of $\mathfrak{f}$ and $\mathfrak{g}$ where $p = q \ge 2$. In the same way we can define $\bar{N}_L(r,a;\ \mathfrak{g})$, $N_E^{1)}(r,a;\ \mathfrak{g})$, $\bar{N}_E^{(2}(r,a;\ \mathfrak{g})$. In a similar manner we can define $\bar{N}_L(r,a;\mathfrak{f})$ and $\bar{N}_L(r,a;\ \mathfrak{g})$ for $a \in \mathbb{C} \cup \{\infty\}$. When $\mathfrak{f}$ and $\mathfrak{g}$ share (a,m), $m \ge 1$,
then $N_E^{1)}(r,a;\mathfrak{f}) = N(r,a;\mathfrak{f}| = 1)$.

Recently, people have raised great interest in difference analogues of Nevanlinna's theory and many articles have focused on value distribution and uniqueness of difference polynomials of entire or meromorphic functions.

In 2013, Liu-Cao-Qi-Yi [8] studied the following theorem.

Theorem A. [8] Let $\mathfrak{f}(z)$ and $\mathfrak{g}(z)$ be two transcendental meromorphic functions of zero order. Suppose that q and c are two non-zero complex constants and $n \in \mathbb{N}$ is such that $\mathfrak{f}^n\mathfrak{f}(qz+c)$ and $\mathfrak{g}^n\mathfrak{g}(qz+c)$ share (1,l).

i. If $l = \infty$ and $n \ge 14$;
ii. If $l = 0$ and $n \ge 26$,

then $\mathfrak{f}(z) \equiv t\mathfrak{g}(z)$ or $\mathfrak{f}(z)\mathfrak{g}(z) \equiv t$ for some constants t that satisfy $t^{n+1} = 1$.

In the same year, Huang [9] studied the analogous result considering q −shift operator for CM sharing while Qi-Yang [10] supplemented the same for IM sharing.

Theorem B. [9,10] Let $\mathfrak{f}(z)$ and $\mathfrak{g}(z)$ be two transcendental meromorphic functions of zero order. Suppose that q is a non-zero complex constant and $n \in \mathbb{N}$ is such that $\mathfrak{f}^n\mathfrak{f}(qz)$ and $\mathfrak{g}^n\mathfrak{g}(qz)$ share $(1, l)$.

i. If $l = \infty$ and $n \geq 14$;
ii. If $l = 0$ and $n \geq 26$,

then $\mathfrak{f}(z) \equiv t\mathfrak{g}(z)$ or $\mathfrak{f}(z)\mathfrak{g}(z) \equiv t$ for some constants t that satisfy $t^{n+1} = 1$.

In 2015, J. P. Wang, Y. Liu and F. H. Liu [3] proved the following theorems and obtained the following results.

Theorem C. [3] Let $c \in \mathbb{C}\backslash\{0\}$ and let $\mathfrak{f}$ and $\mathfrak{g}$ be two transcendental meromorphic functions and $n \geq 14,\ k \geq 3$ be two positive integers. If $E_k(1, \mathfrak{f}^n(z)\mathfrak{f}(z + c) = E_k(1, \mathfrak{g}^n(z)\mathfrak{g}(z + c))$, then $\mathfrak{f} \equiv t_1\mathfrak{g}$ or $\mathfrak{f}\mathfrak{g} \equiv t_2$ for some constant t_1 and t_2 satisfying ${t_1}^{n+1} = 1$ and ${t_2}^{n+1} = 1$.

Theorem D. [3] Let $c \in \mathbb{C}\backslash\{0\}$ and let $\mathfrak{f}$ and $\mathfrak{g}$ be two transcendental meromorphic functions and $n \geq 16, k = 2$ be two positive integers. If $E_2(1, \mathfrak{f}^n(z)\mathfrak{f}(z + c) = E_2(1, \mathfrak{g}^n(z)\mathfrak{g}(z + c))$, then
$\mathfrak{f} \equiv t_1\mathfrak{g}$ or $\mathfrak{f}\mathfrak{g} \equiv t_2$ for some constant t_1 and t_2 satisfying ${t_1}^{n+1} = 1$ and ${t_2}^{n+1} = 1$.

Theorem E. [3] Let $c \in \mathbb{C}\backslash\{0\}$ and let $\mathfrak{f}$ and $\mathfrak{g}$ be two transcendental meromorphic functions and $n \geq 22, k = 1$ be two positive integers. If $E_1(1, \mathfrak{f}^n(z)\mathfrak{f}(z + c) = E_1(1, \mathfrak{g}^n(z)\mathfrak{g}(z + c))$, then
$\mathfrak{f} \equiv t_1\mathfrak{g}$ or $\mathfrak{f}\mathfrak{g} \equiv t_2$ for some constant t_1 and t_2 satisfying ${t_1}^{n+1} = 1$ and ${t_2}^{n+1} = 1$.

Let $P(z) = a_m z^m + a_{m-1}z^{m-1} + \cdots + a_0$ is non-zero polynomial of degree m and $\mathfrak{r}_0 = m_1 + m_2$ and $\mathfrak{r}_1 = m_1 + 2m_2$, where m_1 and m_2 respectively be the number of simple and multiple zeros of $P(z)$.

Regarding Theorems C, D and E , one may ask the following question which is the motivation of the present paper.

Question 1. What can be said about the meromorphic functions $\mathfrak{f}$ and $\mathfrak{g}$ if we consider the difference operator of the form $\Delta_q\mathfrak{f} = \mathfrak{f}(qz + c) - \mathfrak{f}(z)$.

In this paper, we paid our attention to the above question and proved the following two theorems that extend Theorem C, D and Theorem E respectively. Indeed, the following theorems are the main results of the paper.

2 Main Results

Theorem 2.1. Let $\mathfrak{f}(z)$ and $\mathfrak{g}(z)$ be two transcendental meromorphic functions of zero order $c \in \mathbb{C}$, such that $\mathfrak{f}(qz + c) - \mathfrak{f}(z) \not\equiv 0$ and $\mathfrak{g}(qz + c) - \mathfrak{g}(z) \not\equiv 0$, where n, q and c are non zero complex constants. Suppose that $\mathcal{F}(z) = P(\mathfrak{f})(z)\{\mathfrak{f}(qz + c) - \mathfrak{f}(z)\}$ and $\mathcal{G}(z) = P(\mathfrak{g})\{\mathfrak{g}(qz + c) - \mathfrak{g}(z)\}$ share $(1, l)$. Now

1)if $l \geq 2$ and $n > 2\mathfrak{r}_1 + 13$;
2)if $l = 1$ and $n > 2\mathfrak{r}_1 + \frac{\mathfrak{r}_0}{2} + 15$;
3)if $l = 0$ and $n > 2\mathfrak{r}_1 + 3\mathfrak{r}_0 + 25$.

Then one of the following results holds:

$i)$ $P(\mathfrak{f})(z)\{\mathfrak{f}(qz + c) - \mathfrak{f}(z)\}.P(\mathfrak{g})\{\mathfrak{g}(qz + c) - \mathfrak{g}(z)\} \equiv 1$;
$ii)$ $P(\mathfrak{f})(z)\{\mathfrak{f}(qz + c) - \mathfrak{f}(z)\} \equiv P(\mathfrak{g})\{\mathfrak{g}(qz + c) - \mathfrak{g}(z)\}$;
$iii)$ $\mathfrak{f} \equiv t\mathfrak{g}$, for some constant t such that $t^{n+5} = 1$.

Theorem 2.2. Let $\mathfrak{f}(z)$ and $\mathfrak{g}(z)$ be two transcendental meromorphic functions of zero order $c \in \mathbb{C}$, such that $\mathfrak{f}(qz+c)-\mathfrak{f}(z) \not\equiv 0$ and $\mathfrak{g}(qz+c)-\mathfrak{g}(z) \not\equiv 0$, where n, q and c are non zero complex constants. Suppose that $(\mathcal{F}(z))^{(k)} = [P(\mathfrak{f})(z)\{\mathfrak{f}(qz+c)-\mathfrak{f}(z)\}]^{(k)}$ and $(\mathcal{G}(z))^{(k)} = [P(\mathfrak{g})(z)\{\mathfrak{g}(qz+c)-\mathfrak{g}(z)\}]^{(k)}$ share $(1,l)$. Now

1) if $l \geq 2$ and $n > 2(m_2+1)k + 2\mathfrak{r}_1 + 13$;
2) if $l = 1$ and $n > \left(\frac{5}{2}m_2+3\right)k + 2\mathfrak{r}_1 + \frac{\mathfrak{r}_0}{2} + 15$;
3) if $l = 0$ and $n > 5m_2k + 8k + 2\mathfrak{r}_1 + 3\mathfrak{r}_0 + 25$.

Then one of the following results holds:

$i)$ $[P(\mathfrak{f})(z)\{\mathfrak{f}(qz+c)-\mathfrak{f}(z)\}]^{(k)}[P(\mathfrak{g})(z)\{\mathfrak{g}(qz+c)-\mathfrak{g}(z)\}]^{(k)} \equiv 1$.
$ii)$ $\mathfrak{f} \equiv t\mathfrak{g}$ for some constant t such that $t^d = 1$, where $d = LCM\{\zeta_j : j = 0,1 \dots . n\}$
and

$$\zeta_j = \begin{cases} j+1 & if \quad a_j \neq 0, \\ n+1 & if \quad a_j = 0. \end{cases}$$

$iii)$ $\mathfrak{f}$ and $\mathfrak{g}$ satisfy the algebraic equation $R(\mathfrak{f},\mathfrak{g}) = 0$, where

$$R(\mathfrak{w}_1,\mathfrak{w}_2) = P(\mathfrak{w}_1)\big(\mathfrak{w}_1(qz+c)-\mathfrak{w}_1(z)\big) - P(\mathfrak{w}_2)\big(\mathfrak{w}_2(qz+c)-\mathfrak{w}_2(z)\big).$$

Example 2.1. $\mathfrak{f}(z) = sinz$ and $\mathfrak{g}(z) = cosz$ and $P(z) = (z-1)^6(z+1)^6$. Take $c = \pi, q = 1, k = 0$, then it is easy to verify that $[P(\mathfrak{f})(z)\{\mathfrak{f}(qz+c)-\mathfrak{f}(z)\}]^{(k)}$ and $[P(\mathfrak{g})(z)\{\mathfrak{g}(qz+c)-\mathfrak{g}(z)\}]^{(k)}$ sharing 1- points . Here $\mathfrak{f}(z)$ and $\mathfrak{g}(z)$ satisfy the algebraic equation $R(\mathfrak{f},\mathfrak{g}) = 0$.
$[P(\mathfrak{f})(z)\{\mathfrak{f}(qz+c)-\mathfrak{f}(z)\}] - [P(\mathfrak{g})(z)\{\mathfrak{g}(qz+c)-\mathfrak{g}(z)\}] = 0$

3 Some Lemmas

For two non-constant meromorphic functions $\mathcal{F}$ and $\mathcal{G}$, $\mathcal{H}$ represents the following function.

$$\mathcal{H} = \left(\frac{\mathcal{F}''}{\mathcal{F}'} - \frac{2\mathcal{F}'}{\mathcal{F}-1}\right) - \left(\frac{\mathcal{G}''}{\mathcal{G}'} - \frac{2\mathcal{G}'}{\mathcal{G}-1}\right). \tag{3.1}$$

Lemma 3.1.[10] Let $\mathfrak{f}$ be a zero order meromorphic function and $q \in \mathbb{C}\{0\}, c \in \mathbb{C}$. Then

$m\left(r, \frac{\mathfrak{f}(z)}{\mathfrak{f}(qz+c)}\right) = S(r,\mathfrak{f})$ and $T\big(r,\mathfrak{f}(qz+c)\big) = T(r,\mathfrak{f}) + S(r,\mathfrak{f})$.

Lemma 3.2.[11] Let $\mathfrak{f}$ be a meromorphic function of finite order and $q \in \mathbb{C}\setminus\{0\}$, $c \in \mathbb{C}$. Then

$$N(r,0;\mathfrak{f}(qz+c) \leq N\big(r,0;\mathfrak{f}(z)\big) + S(r,\mathfrak{f}),$$
$$N(r,\infty;\mathfrak{f}(qz+c) \leq N\big(r,\infty;\mathfrak{f}(z)\big) + S(r,\mathfrak{f}),$$
$$\overline{N}(r,0;\mathfrak{f}(qz+c) \leq \overline{N}\big(r,0;\mathfrak{f}(z)\big) + S(r,\mathfrak{f}),$$
$$\overline{N}(r,\infty;\mathfrak{f}(qz+c) \leq \overline{N}\big(r,\infty;\mathfrak{f}(z)\big) + S(r,\mathfrak{f}).$$

Lemma 3.3.[12] Let $\mathcal{F}$ and $\mathcal{G}$ be two non-constant meromorphic functions sharing (1,0) and $\mathcal{H} \not\equiv 0$. Then $N_E^{1)}(r,1;\mathcal{F}) = N_E^{1)}(r,1;\mathcal{G}) \leq N(r,H) + S(r,\mathcal{F}) + S(r,\mathcal{G})$.

Lemma 3.4.[4] If two non-constant meromorphic functions $\mathcal{F}$ and $\mathcal{G}$ share $(1,0)$ and $\mathcal{H} \not\equiv 0$. Then

$$\overline{N}(r,\infty;H) \leq \overline{N}(r,0;\mathcal{F}| \geq 2) + \overline{N}(r,0;\mathcal{G}| \geq 2) + \overline{N}(r,\infty;\mathcal{F}| \geq 2) + \overline{N}(r,\infty;\mathcal{G}| \geq 2) + \overline{N}_*(r,1;\mathcal{F},\mathcal{G}) + \overline{N}_0(r,0;\mathcal{F}') + \overline{N}_0(r,0;\mathcal{G}'),$$

where $\overline{N}_0(r,0;\mathcal{F}')$ we mean the reduced counting function of those zeros of $\mathcal{F}'$ which are not the zeros of $\mathcal{F}(\mathcal{F}-1)$.

Lemma 3.5.[13] Let $\mathfrak{f}$ and $\mathfrak{g}$ be any two non-constant meromorphic functions sharing $(1,l)$ where $0 \le l \le \infty$, then

$$\overline{N}(r,1;\mathfrak{f}) + \overline{N}(r,1;\mathfrak{g}) - N_E^{1)}(r,1;\mathfrak{f}) + \left(l - \tfrac{1}{2}\right)\overline{N}_*(r,1;\mathfrak{f},\mathfrak{g}) \le \tfrac{1}{2}[N(r,1;\mathfrak{f}) + N(r,1;\mathfrak{g})].$$

Lemma 3.6.[14] Let $\mathfrak{f}$ and $\mathfrak{g}$ be any two meromorphic functions and suppose they share $(1,l)$, then

$$\overline{N}_*(r,1;\mathfrak{f},\mathfrak{g}) \le \tfrac{1}{l+1}[\overline{N}(r,0;\mathfrak{f}) + \overline{N}(r,\infty;\mathfrak{f}) + \overline{N}(r,0;\mathfrak{g}) + \overline{N}(r,\infty;g)] + S(r,\mathfrak{f}) + S(r,\mathfrak{g}).$$

Lemma 3.7.[4] Let $\mathfrak{f}$ and $\mathfrak{g}$ be two non-constant meromorphic functions sharing $(1,2)$. Then one of the following cases holds:

i. $T(r,\mathfrak{f}) \le N_2(r,0;\mathfrak{f}) + N_2(r,0;\mathfrak{g}) + N_2(r,\infty;\mathfrak{f}) + N_2(r,\infty;\mathfrak{g}) + S(r,\mathfrak{f}) + S(r,\mathfrak{g})$;
ii. $\mathfrak{f} = \mathfrak{g}$;
iii. $\mathfrak{f}.\mathfrak{g} = 1$.

Lemma 3.8.[15] Let $\mathfrak{f}$ and $\mathfrak{g}$ be two transcendental meromorphic functions sharing $(1,1)$ and $\mathcal{H} \not\equiv 0$. Then $T(r,\mathfrak{f}) \le N_2(r,0;\mathfrak{f}) + N_2(r,0;\mathfrak{g}) + N_2(r,\infty;\mathfrak{f}) + N_2(r,\infty;\mathfrak{g}) + \tfrac{1}{2}\overline{N}(r,0;\mathfrak{f}) + \tfrac{1}{2}\overline{N}(r,\infty;\mathfrak{f}) + S(r,\mathfrak{f}) + S(r,\mathfrak{g})$.

Lemma 3.9.[15] Let $\mathfrak{f}$ and $\mathfrak{g}$ be two transcendental meromorphic functions sharing $(1,0)$ and $\mathcal{H} \not\equiv 0$. Then

$$T(r,\mathfrak{f}) \le N_2(r,0;\mathfrak{f}) + N_2(r,0;\mathfrak{g}) + N_2(r,\infty;\mathfrak{f}) + N_2(r,\infty;\mathfrak{g}) + 2\overline{N}(r,0;\mathfrak{f}) + 2\overline{N}(r,\infty;\mathfrak{f}) + 2\overline{N}(r,0;\mathfrak{g}) + 2\overline{N}(r,\infty;\mathfrak{g}) + S(r,\mathfrak{f}) + S(r,\mathfrak{g}).$$

Lemma 3.10.[16] Let $\mathfrak{f}$ be a non-constant meromorphic function and let p and k be two positive integers. Then

$$N_p\left(r,\tfrac{1}{\mathfrak{f}^{(k)}}\right) \le T\left(r,\mathfrak{f}^{(k)}\right) - T(r,\mathfrak{f}) + N_{p+k}\left(r,\tfrac{1}{\mathfrak{f}}\right) + S(r,\mathfrak{f});$$
$$N_p\left(r,\tfrac{1}{\mathfrak{f}^{(k)}}\right) \le k\overline{N}(r,\mathfrak{f}) + N_{p+k}\left(r,\tfrac{1}{\mathfrak{f}}\right) + S(r,\mathfrak{f}).$$

Lemma 3.11.[17] Let $\mathfrak{f}(z)$ be non-constant meromorphic function, then

$$T\left(r,P_n(\mathfrak{f})\right) = nT(r,\mathfrak{f}) + S(r,\mathfrak{f}).$$

Lemma 3.12. Let $\mathfrak{f}(z)$ be a transcendental meromorphic function of finite order and let $\mathcal{F} = P(\mathfrak{f})\mathfrak{f}(qz+c) - \mathfrak{f}(z)$, where n is a positive integer. Then $(n-1)T(r,\mathfrak{f}) + S(r,\mathfrak{f}) \le T(r,\mathcal{F})$.

Proof . From First fundamental theorem , Lemmas 3.1 and 3.11, we obtain

$$\begin{aligned}(n+1)T(r,\mathfrak{f}) = T\left(r,\mathfrak{f}(z)P(\mathfrak{f})\right) + S(r,f) &\le T\left(r,\frac{\mathfrak{f}(z)\mathcal{F}}{\mathfrak{f}(qz+c)-\mathfrak{f}(z)}\right) + S(r,\mathfrak{f})\\ &\le T(r,\mathcal{F}) + T\left(r,\tfrac{\mathfrak{f}(qz+c)-\mathfrak{f}(z)}{\mathfrak{f}(z)}\right) + S(r,\mathfrak{f})\\ &\le T(r,\mathcal{F}) + m\left(r,\tfrac{\mathfrak{f}(qz+c)}{\mathfrak{f}(z)}\right) + N\left(r,\tfrac{\mathfrak{f}(qz+c)}{\mathfrak{f}(z)}\right) + S(r,\mathfrak{f})\\ &\le T(r,\mathcal{F}) + 2T(r,\mathfrak{f}) + S(r,\mathfrak{f}).\end{aligned}$$

Therefore $(n-1)T(r,\mathfrak{f}) + S(r,\mathfrak{f}) \le T(r,\mathcal{F})$.

4 Proof of the Theorems

Proof of the Theorem 2.1. Here we consider $\mathcal{F}(z) = P(\mathfrak{f})(z)\{\mathfrak{f}(qz+c) - \mathfrak{f}(z)\}$ and $\mathcal{G}(z) = P(\mathfrak{g})\{\mathfrak{g}(qz+c) - \mathfrak{g}(z)\}$. Then $\mathcal{F}$ and $\mathcal{G}$ share $(1,l)$

Case 1. Let $\mathcal{H} \not\equiv 0$. Using Lemmas 3.3 and 3.4, we have

$$N_E^{1)}(r,1;\mathcal{F}) \leq N(r,\mathcal{H}) + S(r,\mathcal{F}) + S(r,\mathcal{G})$$
$$\leq \overline{N}(r,0;\ \mathcal{F}| \geq 2) + \overline{N}(r,0;\mathcal{G}\,| \geq 2) + \overline{N}(r,\infty;\ \mathcal{F}| \geq 2) + \overline{N}(r,\infty;\ \mathcal{G}| \geq 2)$$
$$+ \overline{N}_0(r,0;\mathcal{F}') + \overline{N}_0(r,0;\mathcal{G}') + \overline{N}_*(r,1;\mathcal{F},\mathcal{G}). \tag{4.1}$$

By the Second Fundamental Theorem. We get

$$T(r,\mathcal{F}) \leq \overline{N}\ (r,0;\mathcal{F}) + \overline{N}(r,\infty;\mathcal{F}) + \overline{N}(r,1;\mathcal{F}) - \overline{N}_0(r,0;\mathcal{F}') + S(r,\mathfrak{f}) \tag{4.2}$$

and

$$T(r,\mathcal{G}) \leq \overline{N}\ (r,0;\mathcal{G}) + \overline{N}(r,\infty;\mathcal{G}) + \overline{N}(r,1;\mathcal{G}) - \overline{N}_0(r,0;\mathcal{G}') + S(r,\mathfrak{g}) \tag{4.3}$$

Combining (4.1),(4.2) and (4.3) with the help of Lemmas 3.5 and 3.6, we have

$$[T(r,\mathcal{F}) + T(r,\mathcal{G})]$$

$$\leq [\overline{N}\ (r,0;\mathcal{F}) + \overline{N}\ (r,0;\mathcal{G})] + [\overline{N}(r,\infty;\mathcal{F}) + \overline{N}(r,\infty;\mathcal{G})]$$
$$+[\ \overline{N}\ (r,1;\mathcal{F}) + \overline{N}\ (r,1;\mathcal{G})] - [\ \overline{N}_0(r,0;\mathcal{F}') + \overline{N}_0(r,0;\mathcal{G}') + S(r,\mathfrak{f}) + \ S(r,\mathfrak{g})$$
$$\leq\ N_2(r,0;\mathcal{F}) + N_2(r,0;\mathcal{G}) + N_2(r,\infty;\mathcal{G}) + N_2(r,\infty;\mathcal{F}) + \left[\overline{N}\ (r,1;\mathcal{F}) + \overline{N}\ (r,1;\mathcal{G}) -\right.$$
$$\left.N_E^{1)}(r,1;\mathcal{F})\right] + \overline{N}_*(r,1;\mathcal{F},\mathcal{G}) + S(r,\mathfrak{f}) + S(r,\mathfrak{g})$$
$$\leq N_2(r,0;\mathcal{F}) + N_2(r,0;\mathcal{G}) + N_2(r,\infty;\mathcal{G}) + N_2(r,\infty;\mathcal{F}) + \tfrac{1}{2}[\ T(r,\mathcal{F}) + T(r,\mathcal{G})] -$$
$$\left(l - \tfrac{3}{2}\right)\overline{N}_*(r,1;\mathcal{F},\mathcal{G}) + S(r,\mathfrak{f}) + S(r,\mathfrak{g})$$
$$\leq\ \tfrac{1}{2}\,[T(r,\mathcal{F}) + T(r,\mathcal{G})] + N_2(r,0;\mathcal{F}) + N_2(r,0;\mathcal{G}) + N_2(r,\infty;\mathcal{G}) + N_2(r,\infty;\mathcal{F}) +$$
$$\tfrac{(3-2l)}{2(l+1)}[\overline{N}\ (r,0;\mathcal{F}) + \overline{N}(r,\infty;\mathcal{F}) + \overline{N}\ (r,0;\mathcal{G}) + \overline{N}(r,\infty;\mathcal{G})] + S(r,\mathfrak{f}) + S(r,\mathfrak{g})\ . \tag{4.4}$$

which implies

$$[T(r,\mathcal{F}) + T(r,\mathcal{G})] \leq 2[N_2(r,0;\mathcal{F}) + N_2(r,0;\mathcal{G}) + N_2(r,\infty;\mathcal{G}) + N_2(r,\infty;\mathcal{F})] + \frac{(3-2l)}{2(l+1)}$$
$$[\overline{N}\ (r,0;\mathcal{F}) + \overline{N}(r,\infty;\mathcal{G}) + \overline{N}\ (r,\infty;\mathcal{F}) + \overline{N}\ (r,\infty;\mathcal{G})] +$$
$$+ S(r,\mathfrak{f}) + S(r,\mathfrak{g}). \tag{4.5}$$

Subcase 1.1. While $l \geq 2$, in view of Lemmas 3.2 and 3.12, from (4.5) we get

$$(n-1)[T(r,\mathfrak{f}) + T(r,\mathfrak{g})] \leq 2[(m_1 + 2m_2)T(r,\mathfrak{f}) + N\big(r,0;\mathfrak{f}(qz+c) - \mathfrak{f}(z)\big) + (m_1 + 2m_2)T(r,\mathfrak{g})$$
$$+N\big(r,0;\mathfrak{g}(qz+c) - \mathfrak{g}(z)\big) + 2\overline{N}(r,\infty;\mathfrak{f}) + N\big(r,\infty;\ \mathfrak{f}(qz+c) - \mathfrak{f}(z)\big) +$$
$$+2\overline{N}(r,\infty;\mathfrak{g}) + N(r,\infty;\mathfrak{g}(qz+c) - \mathfrak{g}(z))] + S(r,\mathfrak{f}) + S(r,\mathfrak{g}).$$
$$\leq 2[(m_1 + 2m_2 + 6)]\{T(r,\mathfrak{f}) + T(r,\mathfrak{g})\} + S(r,\mathfrak{f}) + S(r,\mathfrak{g}). \tag{4.6}$$

From (4.6) it follows that

$$(n-1)[T(r,\mathfrak{f}) + T(r,\mathfrak{g})] \leq [(2\mathfrak{r}_1 + 12)]\{T(r,\mathfrak{f}) + T(r,\mathfrak{g})\} + S(r,\mathfrak{f}) + S(r,\mathfrak{g}),$$

which is a contradiction for $n > 2\mathfrak{r}_1 + 13$.

Subcase 1.2. While $l = 1$, in view of Lemmas 3.2 and 3.12, from (4.5) we get

$$(n-1)[T(r,\mathfrak{f}) + T(r,\mathfrak{g})] \leq 2[(m_1 + 2m_2)T(r,\mathfrak{f}) + N\big(r,0;\mathfrak{f}(qz+c) - \mathfrak{f}(z)\big) + (m_1 + 2m_2)T(r,\mathfrak{g})$$
$$+N\big(r,0;\mathfrak{g}(qz+c) - \mathfrak{g}(z)\big) + 2\overline{N}(r,\infty;\mathfrak{f}) + N\big(r,\infty;\ \mathfrak{f}(qz+c) - \mathfrak{f}(z)\big) +$$
$$+2\overline{N}(r,\infty;\mathfrak{g}) + N(r,\infty;\mathfrak{g}(qz+c) - \mathfrak{g}(z))] + \left(\tfrac{1}{2}\right)[(m_1 + m_2)T(r,\mathfrak{f})$$
$$+N\big(r,0;\mathfrak{f}(qz+c) - \mathfrak{f}(z)\big) + 2\overline{N}(r,\infty;\mathfrak{f}) + (m_1 + m_2)T(r,\mathfrak{g}) +$$
$$N\big(r,0;\mathfrak{g}(qz+c) - \mathfrak{g}(z)\big) +\ \ 2\overline{N}(r,\infty;\mathfrak{g}) + (S(r,\mathfrak{f}) + S(r,\mathfrak{g})\ .$$

$$\leq \left[2(m_1 + 2m_2 + 6) + \left(\frac{1}{2}\right)(m_1 + m_2 + 4)\right]\{T(r,\mathfrak{f}) + T(r,\mathfrak{g})\} + (S(r,\mathfrak{f}) + S(r,\mathfrak{g}). \quad (4.7)$$

From (4.7), it follows that

$$(n-1)[T(r,\mathfrak{f}) + T(r,\mathfrak{g})] \leq [2\mathfrak{r}_1 + 12 + \frac{\mathfrak{r}_0}{2} + 2]\{T(r,\mathfrak{f}) + T(r,\mathfrak{g})\} + (S(r,\mathfrak{f}) + S(r,\mathfrak{g}).$$

which is a contradiction for $n > 2\mathfrak{r}_1 + \frac{\mathfrak{r}_0}{2} + 15$.

Subcase 1.3. Let $l = 0$. Again using Lemmas 3.2 and 3.12, from (4.5) we get

$$(n-1)[T(r,\mathfrak{f}) + T(r,\mathfrak{g})] \leq [2(m_1 + 2m_2 + 6) + 3(m_1 + m_2 + 4)\{T(r,\mathfrak{f}) + T(r,\mathfrak{g})\} + (S(r,\mathfrak{f}) + S(r,\mathfrak{g}). \quad (4.8)$$

From (4.8), we get

$$(n-1)[T(r,\mathfrak{f}) + T(r,\mathfrak{g})] \leq [2\mathfrak{r}_1 + 12 + 3\mathfrak{r}_0 + 12]\{T(r,\mathfrak{f}) + T(r,\mathfrak{g})\} + (S(r,\mathfrak{f}) + S(r,\mathfrak{g}).$$

Which is a contradiction for $n > 2\mathfrak{r}_1 + 3\mathfrak{r}_0 + 25$.

Case 2. Let $\mathcal{H} \equiv 0$, integrating (3.1) we get

$$\frac{1}{\mathcal{F}-1} \equiv \frac{b\mathcal{G}+a-b}{\mathcal{G}-1} \quad (4.9)$$

where $a \neq 0, b$ are constants. From (4.9) it is clear that $\mathcal{F}$ and $\mathcal{G}$ share $(1,\infty)$. Now we consider the following cases.

Subcase 2.1. Let $b \neq 0$ and $a \neq b$. If $b = -1$, from (4.9) we have

$$\mathcal{F} \equiv \frac{-a}{\mathcal{G}-a-1}.$$

From Lemma 3.2, we see that:

$$\overline{N}(r, a+1; \mathcal{G}) = \overline{N}(r,\infty;\mathcal{F}) \leq 2\overline{N}(r,\infty;\mathfrak{f}).$$

So in view of Lemmas 3.2 and 3.12, using second fundamental theorem, we get

$$(n-1)T(r,\mathfrak{g}) \leq \overline{N}(r,0;\mathcal{G}) + \overline{N}(r,\infty;\mathcal{G}) + \overline{N}(r,a+1;\mathcal{G}) + S(r,\mathfrak{g})$$
$$\leq (m_1 + m_2 + 4)T(r,\mathfrak{g}) + 2T(r,\mathfrak{f}) + S(r,\mathfrak{g}).$$

In a similar manner, we can get

$$(n-1)T(r,\mathfrak{f}) \leq (m_1 + m_2 + 4)T(r,\mathfrak{f}) + 2T(r,\mathfrak{g}) + S(r,\mathfrak{f}).$$

Combining the above equations, we can get

$$(n-1)\{T(r,\mathfrak{f}) + T(r,\mathfrak{g})\} \leq (\mathfrak{r}_0 + 6)\{T(r,\mathfrak{f}) + T(r,\mathfrak{g})\} + S(r,\mathfrak{f}) + S(r,\mathfrak{g}).$$

a contradiction for $n > 2\mathfrak{r}_1 + 13$.

If $b \neq -1$, from (4.9) we get

$$\mathcal{F} - \left(1 + \frac{1}{b}\right) = \frac{-a}{b^2\left[\mathcal{G} + \frac{a-b}{b}\right]} \quad .$$

So , $\overline{N}\left(r, \frac{b-a}{b}; \mathcal{G}\right) = \overline{N}(r, \infty; \mathcal{F})$.

Using Lemmas 3.2 and 3.12 and with the same argument as used in the case for $b = -1$, we can get a contradiction.

Subcase 2.2. Let $b \neq 0$ and $a = b$. If $b = -1$ then from (4.9) we have

$$\mathcal{F}\mathcal{G} \equiv 1.$$

that is $P(\mathfrak{f})(z)\{\mathfrak{f}(qz+c) - \mathfrak{f}(z)\}. P(\mathfrak{g})\{\mathfrak{g}(qz+c) - \mathfrak{g}(z)\} \equiv 1$.

In particular, when $P(\mathfrak{f}) = \mathfrak{f}^n$, take $\mathcal{M}(z) = \mathfrak{f}(z)\mathfrak{g}(z)$. When $\mathcal{M}(z)$ is non-constant. We have from the above by using first fundamental theorem and Lemma 3.1, we have

$$T(r, \mathcal{M}^n) = 4T(r, \mathcal{M}) + S(r, \mathcal{M}),$$

a contradiction . So $\mathcal{M}(z)$ must be a constant and $\mathcal{M}(z)^{n+5} \equiv 1$ which implies $\mathfrak{f}\mathfrak{g} \equiv t$ where

$t^{n+5} = 1$.

If $b \neq -1$, from (4.9) we have

$$\frac{1}{\mathcal{F}} \equiv \frac{b\mathcal{G}}{(1+b)\mathcal{G} - 1}$$

Therefore, $\overline{N}\left(r, \frac{1}{1+b}; \mathcal{G}\right) = \overline{N}(r, 0; \mathcal{F})$.

So in view of Lemmas 3.2 and 3.12, using second fundamental theorem, we have

$$\begin{aligned}(n-1)T(r, \mathfrak{g}) &\leq \overline{N}(r, 0; \mathcal{G}) + \overline{N}(r, \infty; \mathcal{G}) + \overline{N}\left(r, \frac{1}{1+b}; \mathcal{G}\right) + S(r, \mathfrak{g}) \\ &\leq (m_1 + m_2 + 4)T(r, \mathfrak{g}) + (m_1 + m_2 + 2)(T(r, \mathfrak{f}) + S(r, \mathfrak{g}).\end{aligned}$$

In a similar manner, we can get

$$(n-1)T(r, \mathfrak{f}) \leq (m_1 + m_2 + 4)T(r, \mathfrak{f}) + (m_1 + m_2 + 2)T(r, \mathfrak{g}) + S(r, \mathfrak{f}).$$

Combining the above equations, we can get

$$(n-1)\{T(r, \mathfrak{f}) + T(r, \mathfrak{g})\} \leq (2\mathfrak{r}_0 + 6)\{T(r, \mathfrak{f}) + T(r, \mathfrak{g})\} + S(r, \mathfrak{f}) + S(r, \mathfrak{g}).$$

a contradiction for $n > 2\mathfrak{r}_1 + 13$.

Subcase 2.3. Let $b = 0$. From (4.9), we obtain

$$\mathcal{F} \equiv \frac{\mathcal{G} + a - 1}{a} \tag{4.10}$$

If $a \neq 1$ then from (4.10) we obtain $\overline{N}(r, 1-a; \mathcal{G}) = \overline{N}(r, 0; \mathcal{F})$.

Now using a similar process as done in Case 2, for $b \neq -1$, we can deduce a contradiction. Therefore $a = 1$ and from (4.10) we obtain $\mathcal{F} \equiv \mathcal{G}$.

That is $P(\mathfrak{f})(z)\{\mathfrak{f}(qz+c) - \mathfrak{f}(z)\} \equiv P(\mathfrak{g})\{\mathfrak{g}(qz+c) - \mathfrak{g}(z)\}$.

This completes the proof.

Proof of the Theorem 2.2.

Let $\Phi = (\mathcal{F}(z))^{(k)} = [P(\mathfrak{f})(z)\{\mathfrak{f}(qz+c) - \mathfrak{f}(z)\}]^{(k)}$ and

$\Psi = (\mathcal{G}(z))^{(k)} = [P(\mathfrak{g})(z)\{\mathfrak{g}(qz+c) - \mathfrak{g}(z)\}]^{(k)}$

Then Φ and Ψ share $(1, l)$. Applying Lemmas 3.1, 3.2 and 3.10 we have

$$N_2(r,0;\Phi) = N_2\big(r,0;(\mathcal{F})^{(k)}\big) \le N_{k+2}(r,0;\mathcal{F}) + k\overline{N}(r,\infty;\mathcal{F}) + S(r,\mathfrak{f}) \le [(m_2+2)k + \mathfrak{r}_1 + 2]T(r,\mathfrak{f}) + S(r,\mathfrak{f}). \quad (4.11)$$

$$N_2(r,\infty;\Phi) = N_2\big(r,\infty;(\mathcal{F})^{(k)}\big) + S(r,\mathfrak{f}) \le N_2(r,\infty;\mathcal{F}) + S(r,\mathfrak{f}) \le 4T(r,\mathfrak{f}) + S(r,\mathfrak{f}). \quad (4.12)$$

$$\overline{N}(r,0;\Phi) = \overline{N}\big(r,0;(\mathcal{F})^{(k)}\big) + S(r,\mathfrak{f}) \le N_{k+1}(r,0;\mathcal{F}) + k\overline{N}(r,\infty;\mathcal{F}) + S(r,\mathfrak{f}) \le [(m_2+2)k + \mathfrak{r}_0 + 2]T(r,\mathfrak{f}) + S(r,\mathfrak{f}). \quad (4.13)$$

$$\overline{N}(r,\infty;\Phi) = \overline{N}\big(r,\infty;(\mathcal{F})^{(k)}\big) + S((r,\mathfrak{f}) \le \overline{N}(r,\infty;\mathcal{F}) + S(r,\mathfrak{f}) \le 2T(r,\mathfrak{f}) + S(r,\mathfrak{f}). \quad (4.14)$$

Here two cases arise.

Case 1. Let $\mathcal{H} \not\equiv 0$. Now by applying Lemma 3.10 we have

$$N_2(r,0;\Phi) \le N_2\big(r,0;(\mathcal{F})^{(k)}\big) + S(r,\mathfrak{f})$$
$$\le T\big(r,(\mathcal{F})^{(k)}\big) - T(r,\mathcal{F}) + N_{k+2}(r,0;\mathcal{F}) + S(r,\mathfrak{f})$$
$$\le T(r,\Phi) - T(r,\mathcal{F}) + N_{k+2}(r,0;\mathcal{F}) + S(r,\mathfrak{f})$$

That is $$T(r,\mathcal{F}) \le T(r,\Phi) - N_2(r,0;\Phi) + N_{k+2}(r,0;\mathcal{F}) + S(r,\mathfrak{f}) \quad (4.15)$$

Combining Lemma 3.12 and (4.15) we have

$$(n-1)T(r,\mathfrak{f}) \le T(r,\mathcal{F}) \le T(r,\Phi) - N_2(r,0;\Phi) + N_{k+2}(r,0;\mathcal{F}) + S(r,\mathfrak{f}). \quad (4.16)$$

Subcase 1.1. While $l \ge 2$, in view of case (i) of Lemma 3.7, using (4.11), (4.12) and (4.16) we have

$$(n-1)T(r,\mathfrak{f}) \le N_2(r,0;\Psi) + N_2(r,\infty;\Phi) + N_2(r,\infty;\Psi) + N_{k+2}(r,0;\mathcal{F}) + S(r,\mathfrak{f}) + S(r,\mathfrak{g}).$$
$$\le (km_2 + \mathfrak{r}_1 + 6)T(r,\mathfrak{f}) + ((m_2+2)k + \mathfrak{r}_1 + 6))T(r,\mathfrak{g}) + S(r,\mathfrak{f}) + S(r,\mathfrak{g}).$$

Similarly

$$(n-1)T(r,\mathfrak{g}) \le (km_2 + \mathfrak{r}_1 + 6)T(r,\mathfrak{g}) + ((m_2+2)k + \mathfrak{r}_1 + 6))T(r,\mathfrak{f}) + S(r,\mathfrak{f}) + S(r,\mathfrak{g}).$$

Combining the above two equations, we have

$$(n-1)\{T(r,\mathfrak{f}) + T(r,\mathfrak{g})\} \le (2km_2 + 2\mathfrak{r}_1 + 2k + 12)\{T(r,\mathfrak{f}) + T(r,\mathfrak{g})\} + S(r,\mathfrak{f}) + S(r,\mathfrak{g}).$$

which is a contradiction for $n > 2(m_2+1)k + 2\mathfrak{r}_1 + 13$.

Subcase 1.2. While $l = 1$, in view of Lemma 3.8, using (4.11),(4.12),(4.13),(4.14) and (4.16) we have

$$(n-1)T(r,\mathfrak{f}) \le N_2(r,0;\Psi) + N_2(r,\infty;\Phi) + N_2(r,\infty;\Psi) + N_{k+2}(r,0;\mathcal{F}) + \frac{1}{2}\overline{N}(r,0;\Phi) + \frac{1}{2}\overline{N}(r,\infty;\Phi) + S(r,\mathfrak{f}) + S(r,\mathfrak{g}).$$

$$\leq [(m_2+2)k+\mathfrak{r}_1+2]T(r,\mathfrak{g})+4T(r,\mathfrak{f})+4T(r,\mathfrak{g})+[(m_1+(k+2)m_2+2)T(r,\mathfrak{f})+\frac{1}{2}[(m_2+2)k+\mathfrak{r}_0+2]T(r,\mathfrak{f})]+\frac{1}{2}(2T(r,\mathfrak{f}))+S(r,\mathfrak{f})+S(r,\mathfrak{g}).$$

Similarly

$$(n-1)T(r,\mathfrak{g}) \leq [(m_2+2)k+\mathfrak{r}_1+2]T(r,\mathfrak{f})+4T(r,\mathfrak{g})+4T(r,\mathfrak{f})+[(m_1+(k+2)m_2+2)T(r,\mathfrak{g})+\frac{1}{2}[(m_2+2)k+\mathfrak{r}_0+2]T(r,\mathfrak{g})]+\frac{1}{2}(2T(r,\mathfrak{g}))+S(r,\mathfrak{f})+S(r,\mathfrak{g}).$$

Combining the above two equations, we have

$$(n-1)[T(r,\mathfrak{f})+T(r,\mathfrak{g})] \leq \left[\left(\frac{5}{2}m_2+3\right)k+2\mathfrak{r}_1+\frac{\mathfrak{r}_0}{2}+14\right][T(r,\mathfrak{f})+T(r,\mathfrak{g})]+S(r,\mathfrak{f})+S(r,\mathfrak{g}).$$

which is a contradiction for $n > \left(\frac{5}{2}m_2+3\right)k+2\mathfrak{r}_1+\frac{\mathfrak{r}_0}{2}+15$.

Subcase 1.3. While $l=0$, in view of Lemma 3.9, using (4.11),(4.12),(4.13),(4.14) and (4.16) we have

$$(n-1)T(r,\mathfrak{f}) \leq N_2(r,0;\Psi)+N_2(r,\infty;\Phi)+N_2(r,\infty;\Psi)+N_{k+2}(r,0;\mathcal{F})+2\overline{N}(r,0;\Phi)+2\overline{N}(r,\infty;\Phi)+\overline{N}(r,0;\Psi)+\overline{N}(r,\infty;\Psi)+S(r,\mathfrak{f})+S(r,\mathfrak{g}).$$
$$\leq [(m_2+2)k+\mathfrak{r}_1+2]T(r,\mathfrak{g})+4T(r,\mathfrak{f})+4T(r,\mathfrak{g})+(m_1+(k+2)m_2+2)T(r,\mathfrak{f})+2[(m_2+2)k+\mathfrak{r}_0+2]T(r,\mathfrak{f})+2[2T(r,\mathfrak{f})]+[(m_2+2)k+\mathfrak{r}_0+2]T(r,\mathfrak{g})+2T(r,\mathfrak{g})+S(r,\mathfrak{f})+S(r,\mathfrak{g}).$$

Similarly

$$(n-1)T(r,\mathfrak{g}) \leq [(m_2+2)k+\mathfrak{r}_1+2]T(r,\mathfrak{f})+4T(r,\mathfrak{g})+4T(r,\mathfrak{f})+(m_1+(k+2)m_2+2)T(r,\mathfrak{g})+2[(m_2+2)k+\mathfrak{r}_0+2]T(r,\mathfrak{g})+2[2T(r,\mathfrak{g})]+[(m_2+2)k+\mathfrak{r}_0+2]T(r,\mathfrak{f})+2T(r,\mathfrak{f})+S(r,\mathfrak{f})+S(r,\mathfrak{g}).$$

Combining the above two equations, we have

$(n-1)[T(r,\mathfrak{f})+T(r,\mathfrak{g})] \leq 5m_2k+8k+2\mathfrak{r}_1+3\mathfrak{r}_0+24.$
which is a contradiction for $n > 5m_2k+8k+2\mathfrak{r}_1+3\mathfrak{r}_0+25$.

Case 2. Let $\mathcal{H} \equiv 0$. By integration, we get

$$\frac{1}{\Phi-1} \equiv \frac{b\Psi+a-b}{\Psi-1} \tag{4.17}$$

where $a \neq 0, b$ are constants. From (4.17) it is clear that Φ and Ψ share $(1,\infty)$. Now we consider the following cases.

Subcase 2.1. Let $b \neq 0$ and $a \neq b$. If $b=-1$, then from (4.17) we have

$$\Phi \equiv \frac{-a}{\Psi-a-1}.$$

From Lemma 3.2 and (4.14) it is clear that

$$\overline{N}(r,a+1;\Psi)=\overline{N}(r,\infty;\Phi) \leq 2\overline{N}(r,\infty;\mathfrak{f}).$$

Now using second fundamental theorem we get

$$T(r,\Psi) \le \overline{N}(r,0;\Psi) + \overline{N}(r,\infty;\Psi) + \overline{N}(r,a+1;\Psi) + S(r,\mathfrak{g})$$
$$\le \overline{N}(r,0;\Psi) + \overline{N}(r,\infty;\Psi) + \overline{N}(r,\infty;\Phi) + S(r,\mathfrak{f}) + S(r,\mathfrak{g}).$$

By Lemma 3.10, we see that

$$\overline{N}(r,0;\Psi) \le T(r,\Psi) - T(r,\mathcal{G}) + N_{k+1}(r,0;\mathcal{G}) + S(r,\mathfrak{g}).$$

These two inequalities imply

$$T(r,\mathcal{G}) \le \overline{N}(r,\infty;\Psi) + \overline{N}(r,\infty;\Phi) + N_{k+1}(r,0;\mathcal{G}) + S(r,\mathfrak{f}) + S(r,\mathfrak{g}).$$

From the above equation, using (4.14) and Lemmas 3.2, 3.12 we have

$$(n-1)T(r,\mathfrak{g}) \le \overline{N}(r,\infty;\Psi) + \overline{N}(r,\infty;\Phi) + N_{k+1}(r,0;\mathcal{G}) + S(r,\mathfrak{f}) + S(r,\mathfrak{g}).$$
$$\le 2T(r,\mathfrak{f}) + 2T(r,\mathfrak{g}) + (m_1 + (k+1)m_2 + 2)T(r,\mathfrak{g}) + S(r,\mathfrak{f}) + S(r,\mathfrak{g})$$
$$\le 2T(r,\mathfrak{f}) + (\mathfrak{r}_1 + km_2 + 4)T(r,\mathfrak{g}) + S(r,\mathfrak{f}) + S(r,\mathfrak{g}).$$

As Φ and Ψ are interchangeable, in a similar manner we can get

$$(n-1)T(r,\mathfrak{f}) \le 2T(r,\mathfrak{g}) + (\mathfrak{r}_1 + km_2 + 4)T(r,\mathfrak{f}) + S(r,\mathfrak{f}) + S(r,\mathfrak{g}).$$

Combining above two, we can get

$(n-1)\{T(r,\mathfrak{f}) + T(r,\mathfrak{g})\} \le (\mathfrak{r}_1 + km_2 + 6)\{T(r,\mathfrak{f}) + T(r,\mathfrak{g})\} + S(r,\mathfrak{f}) + S(r,\mathfrak{g})$, a contradiction for $n > 2(m_2+1)k + 2\mathfrak{r}_1 + 13$.

If $b \ne -1$, from (4.17) we obtain that

$$\Phi - \left(1 + \frac{1}{b}\right) \equiv \frac{-a}{b^2\left[\Psi + \frac{a-b}{b}\right]} .$$

So , $$\overline{N}\left(r,\frac{b-a}{b};\Psi\right) \equiv \overline{N}(r,\infty;\Phi).$$

Using Lemmas 3.2, 3.10 and 3.12 with the same argument as used in the case for $b = -1$, we can get a contradiction.

Subcase 2.2. Let $b \ne 0$ and $a = b$. If $b = -1$, then from (4.17) we have
$$\Phi\Psi \equiv 1,$$

That is $[P(\mathfrak{f})(z)\{\mathfrak{f}(qz+c) - \mathfrak{f}(z)\}]^{(k)}[P(\mathfrak{g})(z)\{\mathfrak{g}(qz+c) - \mathfrak{g}(z)\}]^{(k)} \equiv 1$.

If $b \ne -1$, from (4.17) we have

$$\frac{1}{\Phi} \equiv \frac{b\Psi}{(1+b)\Psi - 1} .$$

Therefore, $$\overline{N}\left(r,\frac{1}{1+b};\Psi\right) = \overline{N}(r,0;\Phi).$$

So using the second fundamental theorem, we get

$$T(r,\Psi) \le \overline{N}(r,0;\Psi) + \overline{N}(r,\infty;\Psi) + \overline{N}\left(r,\frac{1}{1+b};\Psi\right) + S(r,\mathfrak{g})$$
$$\le \overline{N}(r,0;\Psi) + \overline{N}(r,\infty;\Psi) + \overline{N}(r,0;\Phi) + S(r,\mathfrak{f}) + S(r,\mathfrak{g}).$$

By Lemma 3.10, we see

$$\overline{N}(r,0;\Psi) \le T(r,\Psi) - T(r,\mathcal{G}) + N_{k+1}(r,0;\mathcal{G}) + S(r,\mathfrak{g}).$$

These two inequalities imply

$$T(r,\mathcal{G}) \le \overline{N}(r,\infty;\Psi) + \overline{N}(r,0;\Phi) + N_{k+1}(r,0;\mathcal{G}) + S(r,\mathfrak{f}) + S(r,\mathfrak{g}).$$

From the above equation, using (4.13), (4.14) and Lemmas 3.2, 3.12, we have

$$\begin{aligned}(n-1)T(r,\mathfrak{g}) &\le \overline{N}(r,\infty;\Psi) + \overline{N}(r,0;\Phi) + N_{k+1}(r,0;\mathcal{G}) + S(r,\mathfrak{f}) + S(r,\mathfrak{g}).\\ &\le [(m_2+2)k + \mathfrak{r}_0 + 2]T(r,\mathfrak{f}) + 2T(r,\mathfrak{g}) + (m_1 + (k+1)m_2 + 2)T(r,\mathfrak{g})\\ &\quad + S(r,\mathfrak{f}) + S(r,\mathfrak{g}).\\ &\le [(m_2+2)k + \mathfrak{r}_0 + 2]T(r,\mathfrak{f}) + [km_2 + \mathfrak{r}_1 + 4]T(r,\mathfrak{g}) + +S(r,\mathfrak{f}) +\\ &\quad S(r,\mathfrak{g}).\end{aligned}$$

As Φ and Ψ are symmetric, in a similar manner, we can get

$$(n-1)T(r,\mathfrak{f}) \le [(m_2+2)k + \mathfrak{r}_0 + 2]T(r,\mathfrak{g}) + [km_2 + \mathfrak{r}_1 + 4]T(r,\mathfrak{f}) + S(r,\mathfrak{f}) + S(r,\mathfrak{g})$$

Combining above two , we get

$$(n-1)\{T(r,\mathfrak{f}) + T(r,\mathfrak{g})\} \le [2(m_2+1)k + \mathfrak{r}_1 + \mathfrak{r}_0 + 6]\{T(r,\mathfrak{f}) + T(r,\mathfrak{g})\} + S(r,\mathfrak{f}) + S(r,\mathfrak{g}),$$

a contradiction for $n > 2(m_2+1)k + 2\mathfrak{r}_1 + 13$.

Subcase 2.3. Let $b = 0$. From (4.17), we obtain

$$\Phi \equiv \frac{\Psi - a - 1}{a}. \tag{4.18}$$

$$\overline{N}(r,1-a;\Psi) = \overline{N}(r,0;\Phi).$$

So using the same argument as done in Case 2, for $b \ne -1$, we can similarly deduce a contradiction. Therefore $a = 1$ and from (4.18) we obtain $\Phi = \Psi$, [18-21]

That is $[P(\mathfrak{f})(z)\{\mathfrak{f}(qz+c) - \mathfrak{f}(z)\}]^{(k)} \equiv [P(\mathfrak{g})(z)\{\mathfrak{g}(qz+c) - \mathfrak{g}(z)\}]^{(k)}$.

Integrating we have $P(\mathfrak{f})(z)\{\mathfrak{f}(qz+c) - \mathfrak{f}(z)\} \equiv P(\mathfrak{g})(z)\{\mathfrak{g}(qz+c) - \mathfrak{g}(z)\} + p(z)$

where $p(z)$ is a polynomial of degree at most $k-1$. If $p(z) \not\equiv 0$, then from the second main theorem for the small function and Lemma 3.12 we get [22-26]

$$\begin{aligned}(n-1)T(r,\mathfrak{f}) &\le T(r,\mathcal{F}) + S(r,f)\\ &\le \overline{N}(r,\mathcal{F}) + \overline{N}\left(r,\frac{1}{\mathcal{F}}\right) + \overline{N}\left(r,\frac{1}{\mathcal{G}}\right) + S(r,f)\\ &\le (\mathfrak{r}_0+4)T(r,\mathfrak{f}) + (\mathfrak{r}_0+2)T(r,\mathfrak{g}) + S(r,f).\end{aligned}$$

Similarly $(n-1)T(r,\mathfrak{g}) \le (\mathfrak{r}_0+4)T(r,\mathfrak{g}) + (\mathfrak{r}_0+2)T(r,\mathfrak{f}) + S(r,\mathfrak{g})$

Therefore, $(n-1)[T(r,\mathfrak{f}) + T(r,\mathfrak{g})] \le (2\mathfrak{r}_0+6)[T(r,\mathfrak{f}) + T(r,\mathfrak{g})] + S(r,f) + S(r,\mathfrak{g})$, which by $n > 2(m_2+1)k + 2\mathfrak{r}_1 + 13$ gives a contradiction. Thus $p(z) \equiv 0$ which implies

$$P(\mathfrak{f})(z)\{\mathfrak{f}(qz+c) - \mathfrak{f}(z)\} = P(\mathfrak{g})(z)\{\mathfrak{g}(qz+c) - \mathfrak{g}(z)\}. \tag{4.19}$$

That is $(a_n\mathfrak{f}^n + a_{n-1}\mathfrak{f}^{n-1} + \cdots + a_1\mathfrak{f} + a_0)(\mathfrak{f}(qz+c) - \mathfrak{f}(z)) = (a_n\mathfrak{g}^n + a_{n-1}\mathfrak{g}^{n-1} + \cdots + a_1\mathfrak{g} + a_0)(\mathfrak{g}(qz+c) - \mathfrak{g}(z))$, which implies

$(a_n\mathfrak{f}^n + a_{n-1}\mathfrak{f}^{n-1} + \cdots + a_1\mathfrak{f} + a_0)(\mathfrak{f}(qz+c)) - (a_n\mathfrak{f}^n + a_{n-1}\mathfrak{f}^{n-1} + \cdots + a_1\mathfrak{f} + a_0)\mathfrak{f}(z) =$
$(a_n\mathfrak{g}^n + a_{n-1}\mathfrak{g}^{n-1} + \cdots + a_1\mathfrak{g} + a_0)(\mathfrak{g}(qz+c)) - (a_n\mathfrak{g}^n + a_{n-1}\mathfrak{g}^{n-1} + \cdots + a_1\mathfrak{g} + a_0)\mathfrak{g}(z)$. Let $\mathfrak{h} = \frac{\mathfrak{f}}{\mathfrak{g}}$, we consider the following cases.

Case 1. If $\mathfrak{h}(z)$ is a constant, then substituting $\mathfrak{f} = \mathfrak{g}\mathfrak{h}$ into (4.19), we have

$(a_n\mathfrak{h}^n\mathfrak{g}^n + a_{n-1}\mathfrak{h}^{n-1}\mathfrak{g}^{n-1} + \cdots + a_1\mathfrak{h}\mathfrak{g} + a_0)\mathfrak{h}\mathfrak{g}\big((qz+c) - (a_n\mathfrak{g}^n + a_{n-1}\mathfrak{g}^{n-1} + \cdots + a_1\mathfrak{g} + a_0\big)\mathfrak{g}(qz+c)$
$- \big((a_n\mathfrak{g}^n\mathfrak{h}^n + a_{n-1}\mathfrak{g}^{n-1}\mathfrak{h}^{n-1} + \cdots + a_1\mathfrak{g}\mathfrak{h} + a_0)\mathfrak{h}\mathfrak{g}(z)\big) -$
$(a_n\mathfrak{g}^n + a_{n-1}\mathfrak{g}^{n-1} + \cdots + a_1\mathfrak{g} + a_0)\mathfrak{g}(z)) = 0$, which implies $a_n\mathfrak{g}^n\mathfrak{g}(qz+c)(\mathfrak{h}^{n+1} - 1) +$
$a_{n-1}\mathfrak{g}^{n-1}\mathfrak{g}(qz+c)(\mathfrak{h}^n - 1) + \cdots + a_1\mathfrak{g}^1\mathfrak{g}(qz+c)(\mathfrak{h}^2 - 1) + a_0\mathfrak{g}(qz+c)(\mathfrak{h} - 1) -$
$(a_n\mathfrak{g}^{n+1}(\mathfrak{h}^{n+1} - 1) + a_{n-1}\mathfrak{g}^n(\mathfrak{h}^n - 1) + \cdots + a_1\mathfrak{g}^2(\mathfrak{h}^2 - 1) + a_0\mathfrak{g}((\mathfrak{h} - 1)) = 0.$
Therefore $a_n\mathfrak{g}^n\big(\mathfrak{g}(qz+c) - \mathfrak{g}(z)\big)(\mathfrak{h}^{n+1} - 1) + a_{n-1}\mathfrak{g}^{n-1}\big(\mathfrak{g}(qz+c) - \mathfrak{g}(z)\big)(\mathfrak{h}^n - 1) + \cdots + a_1\mathfrak{g}\big(\mathfrak{g}(qz + c) - \mathfrak{g}(z)\big)(\mathfrak{h}^2 - 1) + a_0\big(\mathfrak{g}(qz+c) - \mathfrak{g}(z)\big)(\mathfrak{h} - 1) = 0$. This implies $\mathfrak{h}^d = 1$, where $d = LCM\{\zeta_j : j = 0,1 \ldots . n\}$ and

$$\zeta_j = \begin{cases} j+1 & if \quad a_j \neq 0, \\ n+1 & if \quad a_j = 0. \end{cases}$$

Thus $\mathfrak{f} \equiv \mathscr{t}\mathfrak{g}$, where $\mathscr{t}$ is a constant with $\mathscr{t}^d = 1$, where $d = LCM\{\zeta_j : j = 0,1 \ldots . n\}$ and

$$\zeta_j = \begin{cases} j+1 & if \quad a_j \neq 0, \\ n+1 & if \quad a_j = 0. \end{cases}$$

Case 2. Suppose $\mathfrak{h}(z)$ is not a constant, then $\mathfrak{f}$ and $\mathfrak{g}$ satisfy the algebraic equation $R(\mathfrak{f}, \mathfrak{g}) = 0$, where $R(\mathfrak{w}_1, \mathfrak{w}_2) = P(\mathfrak{w}_1)\big(\mathfrak{w}_1(qz+c) - \mathfrak{w}_1(z)\big) - P(\mathfrak{w}_2)\big(\mathfrak{w}_2(qz+c) - \mathfrak{w}_2(z)\big)$.

This completes the proof.

5 Conclusions

Using the Nevanlinna theory, we have studied the value distribution and uniqueness of difference operator for a transcendental meromorphic functions $P(\mathfrak{f})(z)\{\mathfrak{f}(qz+c) - \mathfrak{f}(z)\}$ and $P(\mathfrak{g})(z)\{\mathfrak{g}(qz+c) - \mathfrak{g}(z)\}$ having zero order. Our results extends and generalizes the Theorems C, D and E. Also by considering the concept of weighted sharing introduced by Indrajit Lahiri, we have obtained Theorems 2.1 and 2.2 which are the main results of this paper.

Also, we can pose the following open problems.

5.1 Open problems

1. Can the condition for the lower bound n in Theorems 2.1-2.2 be reduced any further?
2. What happens to Theorems 2.1-2.2, if we replace $q-$ shift difference operator $\mathfrak{f}(qz+c) - \mathfrak{f}(z)$ by product of difference operator $\prod_{i=1}^{\eta}(\Delta_{\omega}^{\vartheta}\,\mathfrak{f})^{\mu_i}$, where ϑ, μ_i are positive integers.

5.2 Scope of the research

Nevanlinna theory appeared to be a most powerful tool in investigating analytic solutions of complex differential equations in the complex plane and in the unit disk. In the theory of differential equations, in most of the cases it is difficult to find an explicit solution for a given differential equation. But Nevanlinna theory offers an efficient way for this problem, with the only requirement that the solution must be meromorphic either in the whole complex plane or in small domain where the growth of the solution is sufficiently large near the boundary of the domain. Some of the applications of such results can be seen in signal processing, communication networks, design of filters etc.

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Competing Interests

Authors have declared that no competing interests exist.

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