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Controllability for some nonlinear impulsive partial functional integrodifferential systems with infinite delay in Banach spaces

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Abstract: This work concerns the study of the controllability for some impulsive partial functional integrodifferential equation with infinite delay in Banach spaces. We give sufficient conditions that ensure the controllability of the system by supposing that its undelayed part admits a resolvent operator in the sense of Grimmer, and by making use of the measure of noncompactness and the Mönch fixed-point Theorem. As a result, we obtain a generalization of the work of K. Balachandran and R. Sakthivel (*Journal of Mathematical Analysis and Applications, 255, 447-457, (2001)*) and a host of important results in the literature, without assuming the compactness of the resolvent operator. An example is given for illustration.

Keywords: Controllability, impulsive functional differential equation, infinite delay, resolvent operator, measure of noncompactness, Mönch's fixed-point theorem.

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1. Introduction

The dynamics of evolution processes is often subjected to abrupt changes such as shocks, harvesting, and natural disasters. Often these short-term perturbations are treated as having acted instantaneously or in the form of impulses [1]. The study of dynamical systems with impulsive effects is of great importance. Impulsive differential equations have become more important in recent years in some mathematical models of real processes and phenomena studied in control, physics, chemistry, population dynamics, aero- nautics and engineering. The concept of controllability plays an important role in many areas of applied mathematics. In recent years, significant progress has been made in the controllability of linear and nonlinear deterministic infinite dimensional systems, see for instance [2–11] and the references therein. Many authors studied the controllability problem of nonlinear systems with delay in infinite dimensional Banach spaces; see for instance [2,6,9–11] etc and the references contained in them.

The controllability problem for nonlinear impulsive systems in infinite dimensional Banach spaces has been studied by several authors, see e.g., [6,7,11]. In [11], Selvi and Arjunan considered the following impulsive differential systems with finite delay

$$\begin{cases} x'(t) = A(t)x(t) \;+\; f(t,x_t) + Cu(t), & \text{for } t \in J = [0,b],\ t \neq t_k,\ k = 1,2,\cdots,m, \\ \Delta x(t_k) = I_k(x_{t_k}), & k = 1,2,\cdots,m, \\ x(t) = \varphi(t), & t \in [-r,0]. \end{cases} \tag{1}$$

Using the Hausdorff measure of noncompactness and the Mönch fixed-point theorem and under some sufficient conditions, they obtained a controllability result for Equation (1), without assuming the compactness of the semigroup. In [6], Machado *et al.*, considered the following impulsive mixed-type functional integro-differential system with finite delay and nonlocal conditions of the form

$$\begin{cases} x'(t) = A(t)x(t) + f\left(t, x_t, \int_0^s h(s,t,x_s)ds, \int_0^b k(t,s,x_s)ds\right) + Cu(t) & \text{for } t \in J = [0,b], t \neq t_k, k = 1,2,\cdots,m \\ \Delta x(t_k) = I_k(x_{t_k}), & k = 1,2,\cdots,m \\ x_0 = \phi + g(x), & t \in [-r,0]. \end{cases} \tag{2}$$

Using the Mönch fixed-point theorem via measures of noncompactness and semigroup theory, they obtained a controllability result for Equation (2) without assuming the compactness of the evolution system. However, the result obtained in [11] and [6] are only in connection with finite delay and the impulsive functions I_k $(k = 1, \cdots, m)$ are assumed to be bounded. But since most often many systems arising from realistic models can be described as functional differential and integrodifferential systems with infinite delay [12], it would be natural and interesting to discuss this kind of problems. In an attempt to address this kind of problems, Chang [13] considered the following impulsive functional differential systems with infinite delay;

$$\begin{cases} x'(t) = Ax(t) + f(t, x_t) + Cu(t) & \text{for } t \in J = [0,b],\ t \neq t_k,\ k = 1,2,\cdots,m, \\ \Delta x(t_k) = I_k(x_{t_k}), & k = 1,2,\cdots,m, \\ x(t) = \phi \in \mathcal{BM}_h, \end{cases} \tag{3}$$

where $\mathcal{BM}_h$ is an abstract phase space. Assuming the compactness of the C_0-semigroup generated by A and using Schauder's fixed point theorem together with some sufficient conditions, the author obtained a controllability result for Equation (3).

Motivated by the above works, we study in this paper the controllability for some systems that take the form of the following abstract model of impulsive partial functional integrodifferential equation with infinite delay in a Banach space $(X, \|\cdot\|)$;

$$\begin{cases} x'(t) = Ax(t) + \int_0^t \gamma(t-s)x(s)ds + f(t, x_t) + Cu(t) & \text{for } t \in J = [0,b],\ t \neq t_k,\ k = 1,2,\cdots,m, \\ \Delta x(t_k) = I_k(x_{t_k}), & k = 1,2,\cdots,m \\ x_0 = \varphi \in \mathcal{P}, \end{cases} \tag{4}$$

where $A : \mathcal{D}(A) \to X$ is the infinitesimal generator of a C_0-semigroup $\big(T(t)\big)_{t\geq 0}$ on a Banach space X; for $t \geq 0$, $\gamma(t)$ is a closed linear operator with domain $\mathcal{D}(\gamma(t)) \supset \mathcal{D}(A)$. The control u belongs to $L^2(J, U)$ which is a Banach space of admissible controls, where U is a Banach space. The operator $C \in \mathcal{L}(U, X)$, where $\mathcal{L}(U, X)$ denotes the Banach space of bounded linear operators from U into X, and the phase space $\mathcal{P}$ is a linear space of functions mapping $]-\infty, 0]$ into X satisfying axioms which will be described later, for every $t \geq 0$, x_t denotes the history function of $\mathcal{P}$ defined by $x_t(\theta) = x(t+\theta)$ for $-\infty \leq \theta \leq 0$. Here $0 < t_1 < \cdots < t_m < t_{m+1} < b$ are prefixed numbers, $f : J \times \mathcal{P} \to X$, $I_k : \mathcal{P} \to X$ are appropriate functions satisfying some conditions, and the symbol $\Delta\xi(t)$ represent the jump of the function ξ at t, which is defined by $\Delta\xi(t) = \xi(t^+) - \xi(t^-)$. In the literature devoted to equations with finite delay, the phase space is the space of continuous functions on $[-r, 0]$, for some $r > 0$, endowed with the uniform norm topology. But when the delay is unbounded, the selection of the phase space $\mathcal{P}$ plays an important role in both qualitative and quantitative theories. A usual choice is a normed space satisfying some suitable axioms, which was introduced by Hino *et al.*, [14]. In this work, we use resolvent operators for integral equations, the Mönch fixed-point theorem and the measure of noncompactness, without any compactness assumption on the resolvent operators.

In [15], Grimmer proved the existence and uniqueness of resolvent operators for this type of functional integrodifferential equations that give the variation of parameters formula for the solution. In [16], Desch, Grimmer and Schappacher proved the equivalence of the compactness of the resolvent operator and that of the operator semigroup. In this work, we use the equivalence between the operator-norm continuity of the associated resolvent operator and that of the operator semigroup. This property allows us to drop the compactness assumption on the operator semigroup, considered by the authors in [2,10], and prove that the operator solution satisfies the Mönch condition. The variation of parameters formula for the mild solutions of Equation (4) is given by the resolvent operator, and we prove the controllability result using the Mönch fixed-point theorem and the measure of noncompactness. This method enables us overcome the resolvent operator case considered in this work. In contrary to the evolution semigroup case considered in [6,11], here the semigroup property can not be used because resolvent operators in general do not form semigroups.

To the best of our knowledge, up to now no work has reported on controllability of impulsive partial functional integrodifferential Equation (4) with infinite delay. It has been an untreated topic in the literature, and this fact is the main aim and motivation of the present work.

The work is organized as follows; Section 2 is devoted to preliminary results. In this Section, we give the definition of resolvent operator. This allows us to define the mild solution of Equation (4). In Section 3, we study the controllability of Equation (4). In Section 4, we give an example to illustrate this work.

2. Integrodifferential equations, measure of noncompactness and Mönch's theorem

In this Section, we introduce some definitions and lemmas that will be used throughout the paper. Let $J = [0, b], \ \ b > 0$ and let X be a Banach space. A measurable function $x : J \to X$ is Bochner integrable if and only if $\|x\|$ is Lebesgue integrable. We denote by $L^1(J, X)$ the Banach space of Bochner integrable functions $x : J \to X$ normed by

$$\|x\|_{L^1} = \int_0^b \|x(t)\| dt.$$

In considering the impulsive condition, it is important to introduce some additional concepts and notations. We say that a function $x : [\mu, \eta] \to X$ is a normalized piecewise continuous function on $[\mu, \eta]$ if x is piecewise continuous, and left continuous on $(\mu, \eta]$. Let $\mathcal{PC}([\mu, \eta], X)$ denote the space of normalized piecewise continuous functions from $[\mu, \eta]$ to X. The notation $\mathcal{PC}$ stands for the space of all functions $x : [\mu, \eta] \to X$ such that x is continuous at $t \neq t_k$, $x(t_k^-) = x(t_k)$ and $x(t_k^+)$ exists for all $k = 1, 2, \cdots, m$. In this Section, $(\mathcal{PC}, \| \cdot \|_{\mathcal{PC}})$ is a Banach space endowed with the norm $\|x\|_{\mathcal{PC}} = \sup_{s \in J} \|x(s)\|$.

In this work, we will employ an axiomatic definition of the phase space $\mathcal{P}$ introduced by Hino *et al.*, in [14]. Thus, $(\mathcal{P}, \| \cdot \|_{\mathcal{P}})$ will be a normed linear space of functions mapping $] - \infty, 0]$ into X and satisfying the following axioms;

(A_1) For $\sigma > 0$, if $x :] - \infty, \mu + \sigma] \to X$ is such that $x_\mu \in \mathcal{P}$ and $x|_{[\mu, \mu+\sigma]} \in \mathcal{PC}([\mu, \mu + \sigma]; X)$ then, for every $t \in [\mu, \mu + \sigma]$, the following conditions hold;

There exist positive constant H and functions $K : \mathbb{R}^+ \to [1, \infty)$ continuous and $M : \mathbb{R}^+ \to [1, \infty)$ locally bounded, and all independent of x, such that

(i) $x_t \in \mathcal{P}$,
(ii) $\|x(t)\| \leq H\|x_t\|_{\mathcal{P}}$, which is equivalent to $\|\varphi(0)\| \leq H\|\varphi\|_{\mathcal{P}}$ for every $\varphi \in \mathcal{P}$,
(iii) $\|x_t\|_{\mathcal{P}} \leq K(t - \mu) \sup\limits_{\mu \leq s \leq t} \|x(s)\| + M(t - \mu)\|x_\mu\|_{\mathcal{P}}$.

(A_2) For the function x in A_1, $t \to x_t$ is a $\mathcal{P}$-valued continuous function for $t \in [\mu, \mu + \sigma]$.
(A_3) The space $\mathcal{P}$ is complete.

Consider the following linear homogeneous equation;

$$\begin{cases} x'(t) = Ax(t) + \displaystyle\int_0^t \gamma(t - s)x(s)ds \ \text{ for } \ \ t \geq 0, \\ x(0) = x_0 \in X. \end{cases} \tag{5}$$

where A and $\gamma(t)$ are closed linear operators on a Banach space X. In the sequel, we assume A and $\big(\gamma(t)\big)_{t\geq 0}$ satisfy the following conditions;

(H_1) A is a densely defined closed linear operator in X, hence $\mathcal{D}(A)$ is a Banach space equipped with the graph norm defined by, $|y| = \|Ay\| + \|y\|$ which will be denoted by $(X_1, | \cdot |)$.
(H_2) $\big(\gamma(t)\big)_{t\geq 0}$ is a family of linear operators on X such that $\gamma(t)$ is continuous when regarded as a linear map from $(X_1, | \cdot |)$ into $(X, \| \cdot \|)$ for almost all $t \geq 0$ and the map $t \mapsto \gamma(t)y$ is measurable for all $y \in X_1$ and $t \geq 0$, and belongs to $W^{1,1}(\mathbb{R}^+, X)$. Moreover there is a locally integrable function $b : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\|\gamma(t)y\| \leq b(t)|y|$ and $\left\|\frac{d}{dt}\gamma(t)y\right\| \leq b(t)|y|$.

Remark 1. Note that (H_2) is satisfied in the modelling of Heat Conduction in materials with memory and viscosity. More details can be found in [17].

Let $\mathcal{L}(X)$ be the Banach space of bounded linear operators on X.

Definition 1. [18] A resolvent operator $\left(R(t)\right)_{t\geq 0}$ for Equation (5) is a bounded operator valued function

$$R\,:\,[0,+\infty)\;\longrightarrow\;\mathcal{L}(X)$$

such that

- **(i)** $R(0) = Id_X$ and $\|R(t)\| \leq Ne^{\beta t}$ for some constants N and β.
- **(ii)** For all $x \in X$, the map $t \mapsto R(t)x$ is continuous for $t \geq 0$.
- **(iii)** Moreover for $x \in X_1$, $R(\cdot)x \in \mathcal{C}^1(\mathbb{R}^+; X) \cap \mathcal{C}(\mathbb{R}^+; X_1)$ and $R'(t)x = AR(t)x + \int_0^t \gamma(t-s)R(s)xds = R(t)Ax + \int_0^t R(t-s)\gamma(s)xds$.

Observe that the map defined on $\mathbb{R}^+$ by $t \mapsto R(t)x_0$ solves Equation (5) for $x_0 \in \mathcal{D}(A)$.

Theorem 1. *[15] Assume that (H_1) and (H_2) hold. Then, the linear Equation (5) has a unique resolvent operator $\left(R(t)\right)_{t\geq 0}$.*

Remark 2. In general, the resolvent operator $\left(R(t)\right)_{t\geq 0}$ for Equation (5) does not satisfy the semigroup law, namely, $R(t+s) \neq R(t)R(s)$ for some $t, s > 0$.

We have the following theorem that establishes the equivalence between the operator-norm continuity of the C_0-semigroup and the resolvent operator for integral equations.

Theorem 2. *[5] Let A be the infinitesimal generator of a C_0-semigroup $\left(T(t)\right)_{t\geq 0}$ and let $\left(\gamma(t)\right)_{t\geq 0}$ satisfy (H_2). Then the resolvent operator $\left(R(t)\right)_{t\geq 0}$ for Equation (5) is operator-norm continuous (or continuous in the uniform operator topology) for $t > 0$ if and only if $\left(T(t)\right)_{t\geq 0}$ is operator-norm continuous for $t > 0$.*

Definition 2. Let $u \in L^2(J, U)$ and $\varphi \in \mathcal{P}$. A function $x :]-\infty, b] \to X$ is called a mild solution of equation (4) if $x(t) = \varphi(t)$ for $t \in (-\infty, 0]$, $\Delta x(t_k) = I_k(x_{t_k})$, $k = 1, 2, \cdots, m$, the restriction of x to intervals $J_k = (t_k, t_{k+1}]$ $(k = 0, \cdots, m)$ is continuous and the following integral equation is satisfied

$$x(t) = R(t)\varphi(0) + \int_0^t R(t-s)\left[f(s, x_s) + Cu(s)\right]\,ds \;+ \sum_{0<t_k<t} R(t-t_k)I_k(x_{t_k}) \text{ for } t \in J\,. \tag{6}$$

Definition 3. Equation (4) is said to be controllable on the interval J if for every $\varphi \in \mathcal{P}$ and $x_1 \in X$, there exists a control $u \in L^2(J, U)$ such that a mild solution x of Equation (4) satisfies the condition $x(b) = x_1$.

For proving the main result of the paper we recall some properties of the measure of noncompactness and the Mönch fixed-point theorem.

Definition 4. [19] Let D be a bounded subset of a normed space Y. The Hausdorff measure of noncompactness (shortly MNC) is defined by

$$\beta(D) = \inf\left\{\epsilon > 0 : D\ has\ a\ finite\ cover\ by\ balls\ of\ radius\ less\ than\ \epsilon\right\}.$$

Theorem 3. *[19] Let D, D_1, D_2 be bounded subsets of a Banach space Y. The Hausdorff MNC has the following properties:*

- *(i) If $D_1 \subset D_2$, then $\beta(D_1) \leq \beta(D_2)$, (monotonicity).*
- *(ii) $\beta(D) = \beta(\overline{D})$.*
- *(iii) $\beta(D) = 0$ if and only if D is relatively compact.*

(iv) $\beta(\lambda D) = |\lambda|\beta(D)$ *for any* $\lambda \in \mathbb{R}$, *(Homogeneity).*
(v) $\beta(D_1 + D_2) \leq \beta(D_1) + \beta(D_2)$, *where* $D_1 + D_2 = \{d_1 + d_2 : d_1 \in D_1, d_2 \in D_2\}$, *(subadditivity).*
(vi) $\beta(\{a\} \cup D) = \beta(D)$ *for every* $a \in Y$.
(vii) $\beta(D) = \beta(\overline{co}(D))$, *where* $\overline{co}(D)$ *is the closed convex hull of* D.
(viii) *For any map* $G : \mathcal{D}(G) \subseteq X \to Y$ *which is Lipschitz continuous with a Lipschitz constant* k, *we have* $\beta(G(D)) \leq k\beta(D)$, *for any subset* $D \subseteq \mathcal{D}(G)$.

Let $R_b = \sup\limits_{t\in[0,b]} \|R(t)\|$, $K_b = \sup\limits_{t\in[0,b]} \|K(t)\|$, $M_b = \sup\limits_{t\in[0,b]} \|M(t)\|$. We now state the following useful result for equicontinuous subsets of $\mathcal{C}([a,b];X)$, where X is a Banach space.

Lemma 1. *[19] Let* $M \subset \mathcal{PC}([a,b];X)$ *be bounded and piecewise equicontinuous on* $[a,b]$. *Then* $\beta(M(t))$ *is piecewise continuous for* $t \in [a,b]$ *and* $\beta(M) = \sup\{\beta(M(t)); t \in [a,b]\}$, *where* $M(t) = \{x(t); x \in M\}$.

Lemma 2. *[19] Let* $M \subset \mathcal{C}([a,b];X)$ *be bounded and equicontinuous. Then the set* $\overline{co}(M)$ *is also bounded and equicontinuous.*

To prove the controllability for Equation (4), we need the following results.

Lemma 3. *[4] If* $(u_n)_{n\geq 1}$ *is a sequence of Bochner integrable functions from* J *into a Banach space* Y *with the estimation* $\|u_n(t)\| \leq \mu(t)$ *for almost all* $t \in J$ *and every* $n \geq 1$, *where* $\mu \in L^1(J,\mathbb{R})$, *then the function* $\psi(t) = \beta(\{u_n(t) : n \geq 1\})$ *belongs to* $L^1(J,\mathbb{R}^+)$ *and satisfies the following estimation* $\beta\left(\left\{\int_0^t u_n(s)ds : n \geq 1\right\}\right) \leq 2\int_0^t \psi(s)ds$.

We now state the following nonlinear alternative of Mönch's type for selfmaps, which we shall use in the proof of the controllability of Equation (4).

Theorem 4. *[20](Mönch, 1980) Let* $\mathcal{K}$ *be a closed and convex subset of a Banach space* Z *and* $0 \in \mathcal{K}$. *Assume that* $F : \mathcal{K} \to \mathcal{K}$ *is a continuous map satisfying Mönch's condition, namely,* $D \subseteq \mathcal{K}$ *be countable and* $D \subseteq \overline{co}(\{0\} \cup F(D))$ *implies* $\overline{D}$ *is compact. Then* F *has a fixed point.*

3. Controllability result

In this Section, we give sufficient conditions ensuring the controllability of Equation (4). For that goal, we need to assume that;

(H_3) (i) The following linear operator $W : L^2(J,U) \to X$ defined by $Wu = \int_0^b R(b-s)Cu(s)\,ds$, is surjective so that it induces an isomorphism between $L^2(J,U)\,/_{KerW}$ and X again denoted by W with inverse W^{-1} taking values in $L^2(J,U)\,/_{KerW}$ [21].
(ii) There exists a function $L_W \in L^1(J,\mathbb{R}^+)$ such that for any bounded set $Q \subset X$ we have $\beta((W^{-1}Q)(t)) \leq L_W(t)\beta(Q)$, where β is the Hausdorff MNC.

(H_4) The function $f : J \times \mathcal{P} \longrightarrow X$ satisfies the following two conditions;
(i) $f(\cdot,\varphi)$ is measurable for $\varphi \in \mathcal{P}$ and $f(t,\cdot)$ is continuous for a.e $t \in J$,
(ii) for every positive integer q, there exists a function $l_q \in L^1(J,\mathbb{R}^+)$ such that $\sup\limits_{\|\varphi\|_{\mathcal{P}}\leq q} \|f(t,\varphi)\| \leq l_q(t)$ for a.e. $t \in J$ and $\liminf_{q\to+\infty} \int_0^b \frac{l_q(t)}{q}dt = l < +\infty$,
(iii) there exists a function $h \in L^1(J,\mathbb{R}^+)$ such that for any bounded set $D \subset \mathcal{P}$, $\beta(f(t,D)) \leq h(t)\sup_{-\infty<\theta\leq 0}\beta(D(\theta))$ for a.e. $t \in J$, where $D(\theta) = \{\phi(\theta) : \phi \in D\}$.

(H_5) $I_k : \mathcal{P} \to X$, $k = 1,2,\cdots,m$ are continuous such that;
(i) There are nondecreasing functions $L_k : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\|I_k(x)\| \leq L_k(\|x\|_{\mathcal{P}})$, $k = 1,2,\cdots,m$, $x \in \mathcal{P}$, and $\liminf_{\rho\to+\infty}\frac{L_k(\rho)}{\rho} = \lambda_k < +\infty$, $k = 1,2,\cdots,m$.
(ii) There exist constants $\alpha_k \geq 0$ such that, $\beta(I_k(D)) \leq \alpha_k \sup_{-\infty<\theta\leq 0}\beta(D(\theta))$, $k = 1,2,\cdots,m$, for every bounded subset D of $\mathcal{P}$. $\tau = \left(1 + 2R_bM_2\|L_W\|_{L^1}\right)(2R_b\|h\|_{L^1} + R_b\sum_{k=0}^m \alpha_k) < 1$, where $R_b = \sup\limits_{0\leq t\leq b}\|R(t)\|$ and M_2 is such that $M_2 = \|C\|$.

Theorem 5. *Suppose that hypotheses $(H_3)-(H_5)$ hold and Equation (5) has a resolvent operator $\left(R(t)\right)_{t\geq 0}$ that is continuous in the operator-norm topology for $t>0$. Then Equation (4) is controllable on J provided that*

$$R_b(1+R_bM_2M_3b)K_b\left(l+\sum_{k=1}^{m}\lambda_k\right)<1, \tag{7}$$

where M_3 is such that $M_3=\|W^{-1}\|$.

Proof. Using (H_3) and given an arbitrary function x, we define the control as usual by the following formula;

$$u_x(t) \;=\; W^{-1}\left\{x_1-R(b)\varphi(0)-\int_0^b R(b-s)f(s,x_s)\,ds-\sum_{0<t_k<t}R(b-t_k)I_k(x_{t_k})\right\}(t) \qquad \text{for } t\in I.$$

For each $x\in\mathcal{PC}$ such that $x(0)=\varphi(0)$, we define its extension $\widetilde{x}$ from $]-\infty,b]$ to X as follows

$$\widetilde{x}(t)=\begin{cases} x(t) & \text{if } t\in[0,b],\\ \varphi(t) & \text{if } t\in]-\infty,0].\end{cases}$$

We define the space $E_b=\left\{x:]-\infty,b]\to X \text{ such that } x|_J\in\mathcal{PC} \text{ and } x_0\in\mathcal{P}\right\}$, where where $x|_J$ is the restriction of x to J. We show, by using this control that the operator $P: E_b\to E_b$ defined by

$$(Px)(t)=R(t)\varphi(0)+\int_0^t R(t-s)\left[f(s,\widetilde{x}_s)+Cu_x(s)\right]ds+\sum_{0<t_k<t}R(t-t_k)I_k(x_{t_k}) \quad \text{for } t\in I=[0,b]$$

has a fixed-point. This fixed point is then a mild solution of Equation (4). Observe that $(Px)(b)=x_1$. This means that the control u_x steers the integrodifferential equation from φ to x_1 in time b which implies that the Equation (4) is controllable on J.

For each $\varphi\in\mathcal{P}$, we define the function $y\in\mathcal{PC}$ by $y(t)=R(t)\varphi(0)$ and its extension $\widetilde{y}$ on $]-\infty,0]$ by

$$\widetilde{y}(t)=\begin{cases} y(t) & \text{if } t\in[0,b],\\ \varphi(t) & \text{if } t\in]-\infty,0].\end{cases}$$

For each $z\in\mathcal{PC}$, set $\widetilde{x}(t)=\widetilde{z}(t)+\widetilde{y}(t)$, where $\widetilde{z}$ is the extension by zero of the function z on $]-\infty,0]$. Observe that x satifies (6) if and only if $z(0)=0$ and

$$z(t)=\int_0^t R(t-s)\left[f(s,\widetilde{z}_s+\widetilde{y}_s)+Cu_z(s)\right]ds+\sum_{0<t_k<t}R(t-t_k)I_k(z_{t_k}+\widetilde{y}_{t_k}) \quad \text{for } t\in[0,b],$$

where $u_z(t)=W^{-1}\left\{x_1-R(b)\varphi(0)-\int_0^b R(b-s)f(s,\widetilde{z}_s+\widetilde{y}_s)\,ds-\sum_{0<t_k<t}R(b-t_k)I_k(z_{t_k}+\widetilde{y}_{t_k})\right\}(t)$.

Now let $E_b^0=\left\{z\in E_b \text{ such that } z_0=0\right\}$. Thus E_b^0 is a Banach space provided with the supremum norm. Define the operator $\Gamma: E_b^0\to E_b^0$ by

$$(\Gamma z)(t)=\int_0^t R(t-s)\left[f(s,\widetilde{z}_s+\widetilde{y}_s)+Cu_z(s)\right]ds+\sum_{0<t_k<t}R(t-t_k)I_k(z_{t_k}+\widetilde{y}_{t_k}) \quad \text{for } t\in[0,b].$$

Note that the operator P has a fixed point if and only if Γ has one. So to prove that P has a fixed point, we only need to prove that Γ has one. For each positive number q, let $B_q=\{z\in E_b^0:\|z\|\leq q\}$. Then, for any $z\in B_q$, we have by axiom (A_1) that

$$
\begin{aligned}
\|z_s+y_s\| &\le \|z_s\|_{\mathcal{P}}+\|y_s\|_{\mathcal{P}}\\
&\le K(s)\|z(s)\|+M(s)\|z_0\|_{\mathcal{P}}+K(s)\|y(s)\|+M(s)\|y_0\|_{\mathcal{P}}\\
&\le K_b\|z(s)\|+K_b\|R(t)\|\|\varphi(0)\|+M_b\|\varphi\|_{\mathcal{P}}\\
&\le K_b\|z(s)\|+K_bR_bH\|\varphi\|_{\mathcal{P}}+M_b\|\varphi\|_{\mathcal{P}}\\
&\le K_b\|z(s)\|+\Big(K_bR_bH+M_b\Big)\|\varphi\|_{\mathcal{P}}\\
&\le K_b\,q+\Big(K_bR_bH+M_b\Big)\|\varphi\|_{\mathcal{P}}.
\end{aligned}
$$

Thus, $\|z_s+y_s\|\le K_b\,q+\Big(K_bR_bH+M_b\Big)\|\varphi\|_{\mathcal{P}}=:q'$. We shall prove the theorem in the following steps;
Step 1. We claim that there exists $q>0$ such that $\Gamma(B_q)\subset B_q$. We proceed by contradiction. Assume that it is not true. Then for each positive number q, there exists a function $z^q\in B_q$, such that $\Gamma(z^q)\notin B_q$, *i.e.*, $\|(\Gamma z^q)(t)\|>q$ for some $t\in[0,b]$. Now we have that

$$
\begin{aligned}
q &< \Big\|(\Gamma z^q)(t)\Big\|\\
&\le R_b\int_0^b\Big\|f(s,\tilde z_s^q+\widetilde y_s)\Big\|\,ds+R_b\int_0^b\|Cu_{z^q}(s)\|\,ds+R_b\sum_{k=0}^m L_k(\|z_{t_k}+\widetilde y_{t_k}\|)\\
&\le R_b\int_0^b\Big\|f(s,\tilde z_s^q+\widetilde y_s)\Big\|\,ds+R_b\sum_{k=0}^m L_k(q')\\
&\quad +R_b\int_0^b\Big\|BW^{-1}\Big[x_1-R(b)\varphi(0)-\int_0^b R(b-s)f(s,\tilde z_s^q)\,ds-\sum_{0<t_k<t}R(b-t_k)I_k(z_{t_k}+\widetilde y_{t_k})\Big]\Big\|\,ds\\
&\le bR_bM_2M_3\left(\|x_1\|+R_b\|\varphi(0)\|+R_b\int_0^b\|f(s,\tilde z_s^q)\|\,ds+R_b\sum_{k=0}^m L_k(q')\right)\\
&\quad +R_b\int_0^b\Big\|f(s,\tilde z_s^q+\widetilde y_s)\Big\|\,ds+R_b\sum_{k=0}^m L_k(q')\\
&\le bR_bM_2M_3\left(\|x_1\|+R_bH\|\varphi\|_{\mathcal{B}}+R_b\int_0^b l_{q'}(s)\,ds+R_b\sum_{k=0}^m L_k(q')\right)+R_b\int_0^b l_{q'}(s)\,ds+R_b\sum_{k=0}^m L_k(q'),
\end{aligned}
$$

where $q':=K_b\,q+q_0$, with $q_0:=\Big(K_bR_bH+M_b\Big)\|\varphi\|_{\mathcal{B}}$. Hence

$$
q\le\Big(1+R_bM_2M_3b\Big)\left(R_b\int_0^b l_{q'}(s)\,ds+R_b\sum_{k=0}^m L_k(q')\right)+R_bM_2M_3b\Big(\|x_1\|+R_bH\|\varphi\|_{\mathcal{B}}\Big).
$$

Dividing both sides by q and noting that $q'=K_bq+q_0\to+\infty$ as $q\to+\infty$, we obtain that

$$
1\le\Big(1+R_bM_2M_3b\Big)R_b\left(\frac{\int_0^b l_{q'}(s)\,ds+\sum_{k=0}^m L_k(q')}{q}\right)+\frac{R_bM_2M_3b\Big(\|x_1\|+R_bH\|\varphi\|_{\mathcal{B}}\Big)}{q}
$$

and

$$
\liminf_{q\to+\infty}\left(\frac{\int_0^b l_{q'}(s)\,ds+\sum_{k=0}^m L_k(q')}{q}\right)=\liminf_{q\to+\infty}\left(\frac{\int_0^b l_{q'}(s)\,ds}{q'}+\frac{\sum_{k=0}^m L_k(q')}{q'}\right)\frac{q'}{q}=\left(l+\sum_{k=0}^m\lambda_k\right)K_b.
$$

Thus we have, $1 \leq \Big(1+R_bM_2M_3b\Big)R_b\left(l+\sum_{k=0}^{m}\lambda_k\right)K_b$, and this contradicts (7). Hence for some positive number q, we must have $\Gamma(B_q)\subset B_q$.

Step 2. $\Gamma:\ E_b^0\to E_b^0$ is continuous. In fact let $\Gamma:=\Gamma_1+\Gamma_2$, where

$$(\Gamma_1 z)(t)=\int_0^t R(t-s)f(s,\widetilde{z}_s+\widetilde{y}_s)\,ds+\sum_{k=0}^{m}R(t-t_k)I_k(z_{t_k}+\widetilde{y}_{t_k})\quad\text{and}\quad(\Gamma_2 z)(t)=\int_0^t R(t-s)Cu_z(s)\,ds.$$

Let $\{z^n\}_{n\geq1}\subset E_b^0$ with $z^n\to z$ in E_b^0. Then there exists a number $q>1$ such that $\|z^n(t)\|\leq q$ for all n and $a.e.$ $t\in J$. So $z^n, z\in B_q$. By $(H_4)-(i)$, $f(t,\tilde{z}_t^n+\widetilde{y}_t)\to f(t,\widetilde{z}_t+\widetilde{y}_t)$ for each $t\in[0,b]$. Also, by $(H_5)-(i)$, $I_k(z_{t_k}^n+\widetilde{y}_{t_k})\to I_k(z_{t_k}+\widetilde{y}_{t_k})$ for each $t\in[0,b]$. And by $(H_4)-(ii)$, $\|f(t,\tilde{z}_t^n+\widetilde{y}_t)-f(t,\widetilde{z}_t+\widetilde{y}_t)\|\leq 2l_{q'}(t)$. Then we have

$$\|\Gamma_1 z^n-\Gamma_1 z\|_{\mathcal{P}}\leq R_b\int_0^b\|f(s,\tilde{z}_s^n+\widetilde{y}_s)-f(s,\widetilde{z}_s+\widetilde{y}_s)\|\,ds+R_b\sum_{k=0}^{m}\|I_k(z_{t_k}^n+\widetilde{y}_{t_k})-I_k(z_{t_k}+\widetilde{y}_{t_k})\|\longrightarrow 0,\ as\ n\to+\infty$$

by dominated convergence Theorem. Also we have that

$$\|\Gamma_2 z^n-\Gamma_2 z\|_{\mathcal{P}}\leq R_b^2M_2M_3b\left(\int_0^b\|f(s,\tilde{z}_s^n+\widetilde{y}_s)-f(s,\widetilde{z}_s+\widetilde{y}_s)\|\,ds+\sum_{k=0}^{m}\|I_k(z_{t_k}^n+\widetilde{y}_{t_k})-I_k(z_{t_k}+\widetilde{y}_{t_k})\right)\longrightarrow 0,$$

by dominated convergence Theorem. Thus $\|\Gamma z^n-\Gamma z\|\leq\|\Gamma_1 z^n-\Gamma_1 z\|+\|\Gamma_2 z^n-\Gamma_2 z\|\longrightarrow 0,\ as\ n\to+\infty$. Hence Γ is continuous on E_b^0.

Step 3. $\Gamma(B_q)$ is equicontinuous on $[0,b]$. In fact let $t_1,\ t_2\in J_k,\ \ t_1<t_2$ and $z\in B_q$, we have

$$\begin{aligned}
&\|(\Gamma z)(t_2)-(\Gamma z)(t_1)\|\leq\int_0^{t_1}\|R(t_2-s)-R(t_1-s)\|\|f(s,\widetilde{z}_s+\widetilde{y}_s)+Cu_z(s)\|\,ds\\
&+\sum_{0<t_k<t_1}\|R(t_2-t_k)-R(t_1-t_k)\|\|I_k(z_{t_k}+\widetilde{y}_{t_k})\|+\sum_{t_1\leq t_k<t_2}\|R(t_1-t_k)\|\|I_k(z_{t_k}+\widetilde{y}_{t_k})\|\\
&+\int_{t_1}^{t_2}\|R(t_2-s)\|\|f(s,\widetilde{z}_s+\widetilde{y}_s)+Cu_z(s)\|\,ds\\
&\leq\int_0^{t_1}\|R(t_2-s)-R(t_1-s)\|l_{q'}(s)\,ds\\
&+\int_0^{t_1}\|R(t_2-s)-R(t_1-s)\|M_2M_3\left(\|x_1\|+R_bH\|\varphi\|_{\mathcal{B}}+R_b\int_0^b l_{q'}(\tau)\,d\tau+\sum_{k=0}^{m}L_k(q')\right)ds\\
&+\sum_{0<t_k<t_1}\|R(t_2-t_k)-R(t_1-t_k)\|L_k(q')+R_b\sum_{t_1\leq t_k<t_2}L_k(q')+\int_{t_1}^{t_2}\|R(t_2-s)\|l_{q'}(s)\,ds\\
&+\int_{t_1}^{t_2}\|R(t_2-s)\|M_2M_3\left(\|x_1\|+R_bH\|\varphi\|_{\mathcal{B}}+R_b\int_0^b l_{q'}(\tau)\,d\tau+\sum_{k=0}^{m}L_k(q')\right)ds.
\end{aligned}$$

By the continuity of $\big(R(t)\big)_{t\geq0}$ in the operator-norm toplogy, the dominated convergence Theorem, we conclude that the right hand side of the above inequality tends to zero and independent of z as $t_2\to t_1$. Hence $\Gamma(B_q)$ is equicontinuous on J.

Step 4. We show that the Mönch's condition holds. Suppose that $D\subseteq B_q$ is countable and $D\subseteq\overline{co}(\{0\}\cup\Gamma(D))$. We shall show that $\beta(D)=0$, where β is the Hausdorff MNC. Without loss of generality, we may assume that $D=\{z^n\}_{n\geq1}$. Since Γ maps B_q into an equicontinuous family, $\Gamma(D)$ is also equicontinuous on J. By $(H_3)-(ii)$, $(H_4)-(iii)$ and Lemma 3, we have that

$$\begin{aligned}
&\beta\Big(\{u_{z^n}(t)\}_{n\geq1}\Big)=\beta\left(W^{-1}\left\{x_1-R(b)\varphi(0)-\int_0^b R(t-b)f\Big(s,\{\tilde{z}_s^n+\widetilde{y}_s\}_{n\geq1}\Big)\,ds-\sum_{0<t_k<t}R(b-t_k)I_k(z_{t_k}^n+\widetilde{y}_{t_k})\right\}\right)\\
&\leq L_W(t)\beta\left(\{x_1-R(b)\varphi(0)\}\right)+L_W(t)\beta\left(\left\{\int_0^b R(t-b)f\Big(s,\{\tilde{z}_s^n+\widetilde{y}_s\}_{n\geq1}\Big)\,ds\right\}_{n\geq1}\right)
\end{aligned}$$

$$
\begin{aligned}
&+ L_W(t)\beta\left(\left\{\sum_{0<t_k<t} R(b-t_k)I_k(z^n_{t_k}+\widetilde{y}_{t_k})\right\}_{n\geq 1}\right)\\
&\leq 2R_bL_W(t)\left(\int_0^b h(s)\beta\left(\{\tilde{z}^n_s\}_{n\geq 1}+\{\widetilde{y}_s\}\right)ds\right)+R_bL_W(t)\sum_{k=0}^{m}\beta\left(\left\{I_k(z^n_{t_k}+\widetilde{y}_{t_k})\right\}_{n\geq 1}\right)\\
&\leq 2R_bL_W(t)\left(\int_0^b h(s)\Big[\beta\left(\{\tilde{z}^n_s\}_{n\geq 1}\right)+\beta\Big(\{\widetilde{y}_s\}\Big)\Big]\,ds\right)+R_bL_W(t)\sum_{k=0}^{m}\alpha_k\sup_{-\infty<\theta\leq 0}\beta\left(\left\{z^n_{t_k}+\widetilde{y}_{t_k}\right\}_{n\geq 1}\right)\\
&\leq 2R_bL_W(t)\left(\int_0^b h(s)\beta\left(\{\tilde{z}^n_s\}_{n\geq 1}\right)\,ds\right)+R_bL_W(t)\sum_{k=0}^{m}\alpha_k\sup_{-\infty<\theta\leq 0}\beta\left(\left\{z^n_{t_k}\right\}_{n\geq 1}\right),
\end{aligned}
$$

since $\Big\{\widetilde{y}_s:\ s\in[0,b]\Big\}$is compact, so

$$
\leq 2R_bL_W(t)\left(\int_0^b h(s)\sup_{-\infty<\theta\leq 0}\beta\left(\{\tilde{z}^n_s(\theta)\}_{n\geq 1}\right)\,ds\right)+R_bL_W(t)\sum_{k=0}^{m}\alpha_k\sup_{-\infty<\theta\leq 0}\beta\left(\left\{z^n_{t_k}\right\}_{n\geq 1}\right),
$$

by Lemma 1, since $D=\{z^n\}_{n\geq 1}$ is equicontinuous, we obtain

$$
\leq 2R_bL_W(t)\left(\int_0^b h(s)\,ds\right)\sup_{0\leq t\leq b}\beta\left(\{z^n(t)\}_{n\geq 1}\right)+R_bL_W(t)\sum_{k=0}^{m}\alpha_k\sup_{0\leq\tau_k\leq t_k}\beta\left(\{z^n(\tau_k)\}_{n\geq 1}\right).
$$

This implies that

$$
\begin{aligned}
&\beta\Big(\{(\Gamma z^n)(t)\}_{n\geq 1}\Big)\leq\beta\left(\left\{\int_0^t R(t-s)f(s,\{\tilde{z}^n_s+\widetilde{y}_s\}_{n\geq 1})\,ds\right\}_{n\geq 1}\right)+\beta\left(\left\{\int_0^t R(t-s)u_{z^n}(s)\,ds\right\}_{n\geq 1}\right)\\
&\quad+\beta\left(\left\{\sum_{0<t_k<t}R(b-t_k)I_k(z^n_{t_k}+\widetilde{y}_{t_k})\right\}_{n\geq 1}\right)\\
&\leq 2R_b\left(\int_0^b h(s)\,ds\right)\sup_{0\leq t\leq b}\beta\left(\{z^n(t)\}_{n\geq 1}\right)+R_b\sum_{k=0}^{m}\alpha_k\sup_{0\leq\tau_k\leq t_k}\beta\left(\{z^n(\tau_k)\}_{n\geq 1}\right)\\
&\quad+2R_bM_2\left(\int_0^b L_W(s)\,ds\right)2R_b\left(\int_0^b h(s)\,ds\right)\sup_{0\leq t\leq b}\beta\left(\{z^n(t)\}_{n\geq 1}\right)\\
&\quad+2R_b^2M_2\left(\int_0^b L_W(s)\,ds\right)\sum_{k=0}^{m}\alpha_k\sup_{0\leq\tau_k\leq t_k}\beta\left(\{z^n(\tau_k)\}_{n\geq 1}\right)\\
&\leq 2R_b\|h\|_{L^1}\sup_{0\leq t\leq b}\beta\left(\{z^n(t)\}_{n\geq 1}\right)+R_b\sum_{k=0}^{m}\alpha_k\sup_{0\leq\tau_k\leq t_k}\beta\left(\{z^n(\tau_k)\}_{n\geq 1}\right)\\
&\quad+2R_bM_2\|L_W\|_{L^1}2R_b\|h\|_{L^1}\sup_{0\leq t\leq b}\beta\left(\{z^n(t)\}_{n\geq 1}\right)+2R_b^2M_2\|L_W\|_{L^1}\sum_{k=0}^{m}\alpha_k\sup_{0\leq\tau_k\leq t_k}\beta\left(\{z^n(\tau_k)\}_{n\geq 1}\right).
\end{aligned}
$$

It follows that

$$
\begin{aligned}
&\beta\Big(\Gamma(D)(t)\Big)\leq 2R_b\|h\|_{L^1}\sup_{0\leq t\leq b}\beta\left(D(t)\right)+R_b\sum_{k=0}^{m}\alpha_k\sup_{0\leq t\leq b}\beta\left(D(t)\right)+2R_bM_2\|L_W\|_{L^1}2R_b\|h\|_{L^1}\sup_{0\leq t\leq b}\beta\left(D(t)\right)\\
&\quad+2R_b^2M_2\|L_W\|_{L^1}\sum_{k=0}^{m}\alpha_k\sup_{0\leq t\leq b}\beta\left(D(t)\right)\\
&\leq\left(2R_b\|h\|_{L^1}+R_b\sum_{k=0}^{m}\alpha_k+2R_bM_2\|L_W\|_{L^1}2R_b\|h\|_{L^1}+2R_b^2M_2\|L_W\|_{L^1}\sum_{k=0}^{m}\alpha_k\right)\sup_{0\leq t\leq b}\beta\left(D(t)\right)\\
&\leq\Big(1+2R_bM_2\|L_W\|_{L^1}\Big)\left(2R_b\|h\|_{L^1}+R_b\sum_{k=0}^{m}\alpha_k\right)\sup_{0\leq t\leq b}\beta\left(D(t)\right).
\end{aligned}
$$

Since D and $\Gamma(D)$ are equicontinuous on $[0,b]$ and D is bounded, it follows by Lemma 1 that $\beta\Big(\Gamma(D)\Big) \leq \tau\beta\Big(D\Big)$, where τ is as defined in (H_5). Thus from the Mönch condition, we get that $\beta\Big(D\Big) \leq \beta\Big(\overline{co}(\{0\} \cup \Gamma(D)\Big) = \beta\Big(\Gamma(D)\Big) \leq \tau\beta\Big(D\Big)$, and since $\tau < 1$, this implies $\beta\Big(D\Big) = 0$, which implies that D is relatively compact as desired in B_q and the Mönch condition is satisfied. We conclude by Theorem 4, that Γ has a fixed point z in B_q. Then $x = z + y$ is a fixed point of P in E_b and thus equation (4) is controllable on $[0,b]$. $\Box$

4. Numerical example

Now, we illustrate our main result by the following example.

Example 1. Consider the partial functional integrodifferential system of the form

$$\begin{cases} \dfrac{\partial v}{\partial t}(t,\xi) = \dfrac{\partial^2 v}{\partial \xi^2}(t,\xi) + \displaystyle\int_0^t \zeta'(t-s)\frac{\partial^2 v}{\partial \xi^2}(s,\xi)\,ds & \\ \qquad\qquad + \displaystyle\int_{-\infty}^0 \alpha(\theta) g(t, v(t+\theta,\xi))\,d\theta + \eta u(t,\xi) & \text{for } t \in J = [0,b] \text{ and } \xi \in (0,\pi) \\ v(t,0) = 0 = v(t,\pi) & \text{for } t \in [0,b], \\ v(t_k^+,\xi) - v(t_k^-,\xi) = \displaystyle\int_{-\infty}^{t_k} \mu_k(t_k - s) v(s,\xi) ds, & \xi \in (0,\pi),\ k = 1,2,\cdots,m, \\ v(\theta,\xi) = \phi(\theta,\xi) & \text{for } \theta \in]-\infty,0] \text{and } \xi \in (0,\pi), \end{cases} \tag{8}$$

where $\eta > 0$, $\phi \in \mathcal{P}$, $I_k > 0$, $k = 1,2,\cdots,m$, $u \in L^2((0,\pi))$, $g: [0,1] \times \mathbb{R} \to \mathbb{R}$ is continuous and Lipschitzian with respect to the second variable, the initial data function $\phi : \mathbb{R}^- \times \Omega \to \mathbb{R}$ is a given function, $\alpha : \mathbb{R}^- \to \mathbb{R}$ is continuous, $\alpha \in L^1(\mathbb{R}^-,\mathbb{R})$ and $\zeta \in \mathcal{C}^2([0,b])$ and $\zeta(0) > 0$.

Let $X = L^2(0,\pi)$, and the phase space $\mathcal{P} = \mathcal{PC}_0 \times L^2(\tilde{h}, X)$ ($\tilde{h} :]-\infty,-r] \to \mathbb{R}$ be a positive function), as introduced in [22]. We define $A : \mathcal{D}(A) \subset X \to X$ by

$$\begin{cases} \mathcal{D}(A) = \Big\{ v \in X :\ v \text{ and } v' \text{ are absolutely continuous, } v'' \in X,\ v(0) = v(\pi) = 0 \Big\} \\ Av\ =\ v'' \ \text{ for each } v \in \mathcal{D}(A). \end{cases}$$

Then, $Av = \sum_{n=1}^{\infty} n^2 \langle v, v_n \rangle v$, $v \in \mathcal{D}(A)$, where $v_n(s) = \sqrt{2/\pi}\sin(ns)$, $n = 1,2,3,\cdots$ is the orthogonal set of eigenvectors of A. It is well known that A is the infinitesimal generator of an analytic semigroup $\big(T(t)\big)_{t\geq 0}$ in X as is given by $T(t)v = \sum_{n=1}^{\infty} exp(-n^2 t)\langle v, v_n\rangle v$, $v \in X$. Moreover, $\big(T(t)\big)_{t\geq 0}$ generated by A above, is compact for $t > 0$ and operator-norm continuous for $t > 0$. Then by Theorem 2, the corresponding resolvent operator is operator-norm continuous. Now define

$$x(t)(\xi) \ =\ v(t,\xi),\ \ x'(t)(\xi) \ =\ \frac{\partial v(t,\xi)}{\partial t},\ \ u(t,\xi) = u(t)(\xi),\ \ \varphi(\theta)(\xi) = \phi(\theta,\xi) \ \text{ for } \theta \in]-\infty,0] \text{ and } \xi \in (0,\pi).$$

$$I_k(\varphi)(\xi) = \int_{-\infty}^0 \mu_k(-s)\varphi(s,\xi)ds,\ \ \xi \in (0,\pi),\ k = 1,2,\cdots,m.$$

$$f(t,\psi)(\xi) \ =\ \int_{-\infty}^0 \alpha(\theta) g(t,\psi(\theta)(\xi))\,d\theta \ \text{ for } \theta \in]-\infty,0] \text{ and } \xi \in (0,\pi).$$

Now $C : X \to X$ be defined by $\Big(Cu(t)\Big)(\xi) \ =\ Cu(t)(\xi) \ =\ \eta u 1_\Gamma(t,\xi)$.

$$(\gamma(t)x)(\xi) \ =\ \zeta(t)\Delta v(t,\xi) \ \text{ for } \ t \in [0,b],\ \ x \in \mathcal{D}(A) \text{ and } \ \xi \in (0,\pi).$$

We suppose that $\varphi \in \mathcal{P}$. Then, Equation (8) is then transformed into the following form

$$\begin{cases} x'(t) = Ax(t) \ +\ \displaystyle\int_0^t \gamma(t-s)x(s)ds \ +\ f(t,x_t) + Cu(t) & \text{for } t \in J = [0,b], \\ \Delta x(t_k) = I_k(x_{t_k}), & k = 1,2,\cdots,m, \\ x_0 = \varphi \in \mathcal{P}. \end{cases} \tag{9}$$

Suppose there exists a continuous function $p \in L^1(J;\mathbb{R}^+)$ such that $|g(t,y_1) - g(t,y_2)| \leq p(t)|y_1 - y_2|$ for $t \in J$ and $y_1, y_2 \in \mathbb{R}$ and $g(t,0) = 0$ for $t \in J$. One can see that f is Lipschitz continuous with respect to the second variable and moreover for $\varphi \in \mathcal{P}$, we have we have $\sup_{\|\varphi\|_{\mathcal{B}} \leq q} \left\| f(t,\varphi) \right\| \leq q \|\alpha\| p(t)$. So f satisfies $(H_4) - (i)$ and $(H_4) - (ii)$ with $l_q(t) = q \|\alpha\| p(t)$. Also f satisfies $(H_4) - (iii)$ by condition $(viii)$ of Theorem 3, since f is Lipschitz. Now for $\xi \in (0,\pi)$, the operator W is given by $(Wu)(\xi) = \eta \int_0^b R(b-s)u(s,\xi)\,ds$. Assuming that W satisfies (H_3), then all the conditions of Theorem 5 hold and Equation (9) is controllable.

5. Conclusion

In this work, we have shown the controllability of some impulsive partial functional integrodifferential differential equation with infinite delay in Banach spaces by using the Hausdorff Measure of Noncompactness and the Mönch fixed point theorem. We achieved this without assuming the compactness of the resolvent operator for the associated undelayed part.

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